Problem 1

a) order = 2

b) linear

c) $\left.\begin{array}{l} y = c \\ y' = 0 \\ y'' = 0 \end{array}\right\} \rightarrow \begin{array}{l} 3c - 0 + 0 = 1 \\ c = \frac{1}{3} \end{array}$

$\boxed{y = \frac{1}{3}}$

Problem 2

a) True

b) True

c) False $\rightarrow$ If $y = 0, y' = 0$

$yy' = 10$

$(0)(0) = 10$

$0 = 10 \leftarrow$ NOT TRUE

d) True $\rightarrow$ If $y = 1, y' = 0$

$0 - t^2 \sin(\pi) = 0$

$0 = 0 \leftarrow$ TRUE

Problem 3

$y = t^m$

$y' = mt^{m-1}$

$y'' = m(m-1)t^{m-2}$

$t^2 y'' - 6y = 0$

$t^2 \left(m(m-1)t^{m-2}\right) - 6t^m = 0$

$m(m-1)t^m - 6t^m = 0$

$t^m (m(m-1) - 6) = 0$

$t^m (m^2 - m - 6) = 0$

$\cancel{t^m}(m-3)(m+2) = 0$

$m = 3, \; m = -2$

$\boxed{B}$

Problem 4

a) $\frac{dT}{dt} = k(A-T)$

$A = 27$

$T(0) = 3$

$T(4) = 15$

$\frac{dT}{dt} = k(27-T)$

$\int \frac{1}{27-T}\,dT = \int k\,dt$

$-\ln(27-T) = kt + C$

$\ln(27-T) = -kt + C$

$27 - T = ce^{-kt}$

$T = 27 - ce^{-kt}$

$T(0)=3 \rightarrow 3 = 27 - ce^{0}$

$C = 24$

$T = 27 - 24e^{-kt}$

$T(4)=15 \rightarrow 15 = 27 - 24e^{-4k}$

$-12 = -24e^{-4k}$

$1/2 = e^{-4k}$

$\ln(1/2) = -4k$

$k = \frac{\ln(1/2)}{-4} = \frac{-\ln(2)}{-4} = \frac{\ln(2)}{4}$

$\boxed{T(t) = 27 - 24e^{-\frac{\ln(2)}{4}t}}$

b) $19 = 27 - 24e^{-\frac{\ln(2)}{4}t}$

$\frac{1}{3} = e^{\frac{\ln(2)}{4}t}$

$\ln(1/3) = -\frac{\ln(2)}{4}t \rightarrow t = \frac{4\ln(1/3)}{-\ln(2)} = \frac{-4\ln(3)}{-\ln(2)} = \boxed{\frac{4\ln(3)}{\ln(2)}\text{ min.}}$

Problem 5

4

Use the Integrating Factor Method.

$$y' + \frac{1}{2\sqrt{t}}\, y = 1$$

$p(t) = \frac{1}{2\sqrt{t}}$ $\quad$ $g(t) = 1$

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{1}{2} t^{-1/2}\,dt} = e^{\sqrt{t}}$$

$$y = \frac{1}{e^{\sqrt{t}}}\left[\int e^{\sqrt{t}} \cdot 1 \, dt\right]$$

To evaluate $\int e^{\sqrt{t}}\,dt$ use substitution and I.B.P.

$$\int e^{\sqrt{t}}\,dt \rightarrow \begin{cases} w = \sqrt{t} \\ dw = \frac{1}{2\sqrt{t}}\,dt \\ dt = 2\sqrt{t}\,dw \end{cases}$$

$= \int e^{w}\, 2\sqrt{t}\,dw$

$= \int 2we^{w}\,dw \rightarrow \begin{cases} u = 2w \\ du = 2\,dw \end{cases}$ $\quad$ $v = e^{w}$, $dv = e^{w}\,dw$

$= 2we^{w} - \int 2e^{w}\,dw$

$= 2we^{w} - 2e^{w} + C$

$= 2\sqrt{t}e^{\sqrt{t}} - 2e^{\sqrt{t}} + C$

$$y = \frac{1}{e^{\sqrt{t}}}\left[2\sqrt{t}e^{\sqrt{t}} - 2e^{\sqrt{t}} + C\right]$$

$$y = 2\sqrt{t} - 2 + \frac{c}{e^{\sqrt{t}}}$$

$$y(1) = -2 \rightarrow -2 = 2 - 2 + \frac{c}{e}$$

$$-2 = \frac{c}{e}$$

$$c = -2e$$

$$\boxed{y = 2\sqrt{t} - 2 - \frac{2e}{e^{\sqrt{t}}}}$$

Problem 6

$$y' + \underbrace{2t}_{p}y = \underbrace{e^{-t^2}\cos t + t^3}_{g}$$

$$\mu(t) = e^{\int 2t\,dt} = e^{t^2}$$

$$y = \frac{1}{e^{t^2}}\left[\int e^{t^2}\left(e^{-t^2}\cos t + t^3\right)dt\right]$$

$$y = \frac{1}{e^{t^2}}\left[\int \cos t\,dt + \int t^3 e^{t^2}\,dt\right]$$

$$\int \cos(t)\,dt = \sin t$$

$$\int t^3 e^{t^2}\,dt \longrightarrow$$

Substitution

$$w = t^2$$

$$dw = 2t\,dt$$

$$dt = \frac{dw}{2t}$$

$$\int t^3 e^w \frac{dw}{2t}$$

$$\int \frac{1}{2} t^2 e^w\,dw$$

I.B.P.

$$\frac{1}{2}\int we^w \longrightarrow \quad u = w \quad v = e^w$$

$$du = dw, \quad dv = e^w\,dw$$

$$\frac{1}{2}\left[we^w - \int e^w\,dw\right]$$

$$\frac{1}{2}\left[t^2e^{t^2} - e^{t^2}\right] + c$$

$$y = \frac{1}{e^{t^2}}\left[\sin t + \frac{1}{2}t^2e^{t^2} - \frac{1}{2}e^{t^2} + c\right]$$

$$\boxed{y = \frac{\sin t}{e^{t^2}} + \frac{1}{2}t^2 - \frac{1}{2} + \frac{c}{e^{t^2}}}$$

Problem 7

This is a separable ODE...

$$\frac{dy}{dx} = \frac{2\cos(2x)}{y+1}$$

$$\int (y+1)\,dy = \int 2\cos(2x)\,dx$$

$$\frac{1}{2}y^2 + y = \sin(2x) + C$$

$$y^2 + 2y = 2\sin(2x) + C$$

$$y(0) = 3 \rightarrow \quad (3)^2 + 2(3) = 2\sin(0) + C$$

$$15 = C$$

$$y^2 + 2y \boxed{+1} = 2\sin(2x) + 15 \boxed{+1}$$

$$(y+1)^2 = 2\sin(2x) + 16$$

$$y + 1 = \pm\sqrt{2\sin(2x) + 16}$$

$y = -1 \pm \sqrt{2\sin(2x)+16}$, but since $y(0) = 3$ we only want the positive root

$$\boxed{y = -1 + \sqrt{2\sin(2x) + 16}}$$

Problem 8

a) $\left.\begin{array}{l} y = k \\ y' = 0 \end{array}\right\} \rightarrow$ $0 = k^2 - 2k$

$0 = k(k-2)$

$k = 0,\ k = 2$

$\boxed{\text{constant sol. } \quad y = 0,\ y = 2}$

b) $\dfrac{dy}{dt} = y^2 - 2y$

$\dfrac{1}{y^2 - 2y}\, dy = 1\, dt$

$\int \dfrac{1}{y(y-2)}\, dy = \int 1\, dt$

P.F.D.

$\dfrac{1}{y(y-2)} = \dfrac{A}{y} + \dfrac{B}{y-2}$

$A = -\frac{1}{2}\ ,\quad B = \frac{1}{2}$

$\int \left(\dfrac{1/2}{y-2} - \dfrac{1/2}{y} \right) dy = \int 1\, dt$

$\frac{1}{2} \left[\ln|(y-2)| - \ln|y| \right] = t + C$

$\ln \sqrt{\dfrac{y-2}{y}} = t + C$

$\sqrt{\dfrac{y-2}{y}} = e^{t+C}$

$\sqrt{\dfrac{y-2}{y}} = Ce^{t}$

$\dfrac{y-2}{y} = (Ce^{t})^2$

$\dfrac{y-2}{y} = Ce^{2t}$

$y - 2 = yCe^{2t}$

$y - yCe^{2t} = 2$

$y(1 - Ce^{2t}) = 2$

$\boxed{y = \dfrac{2}{1 - Ce^{2t}}}$

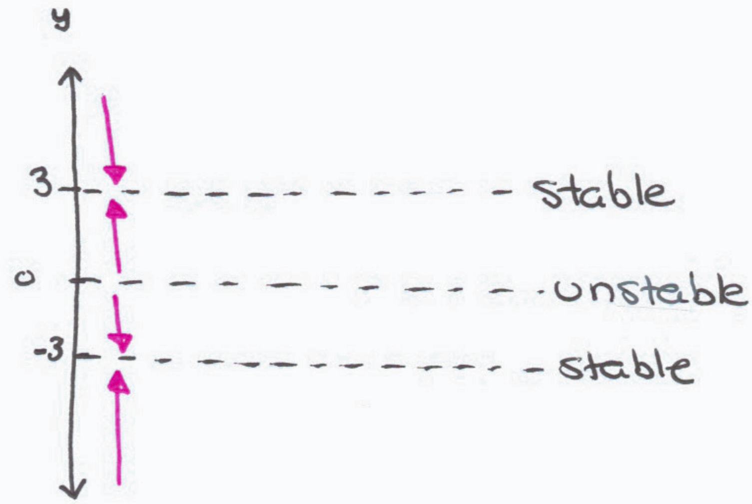
Problem 9

a) $0 = 9y - y^3$

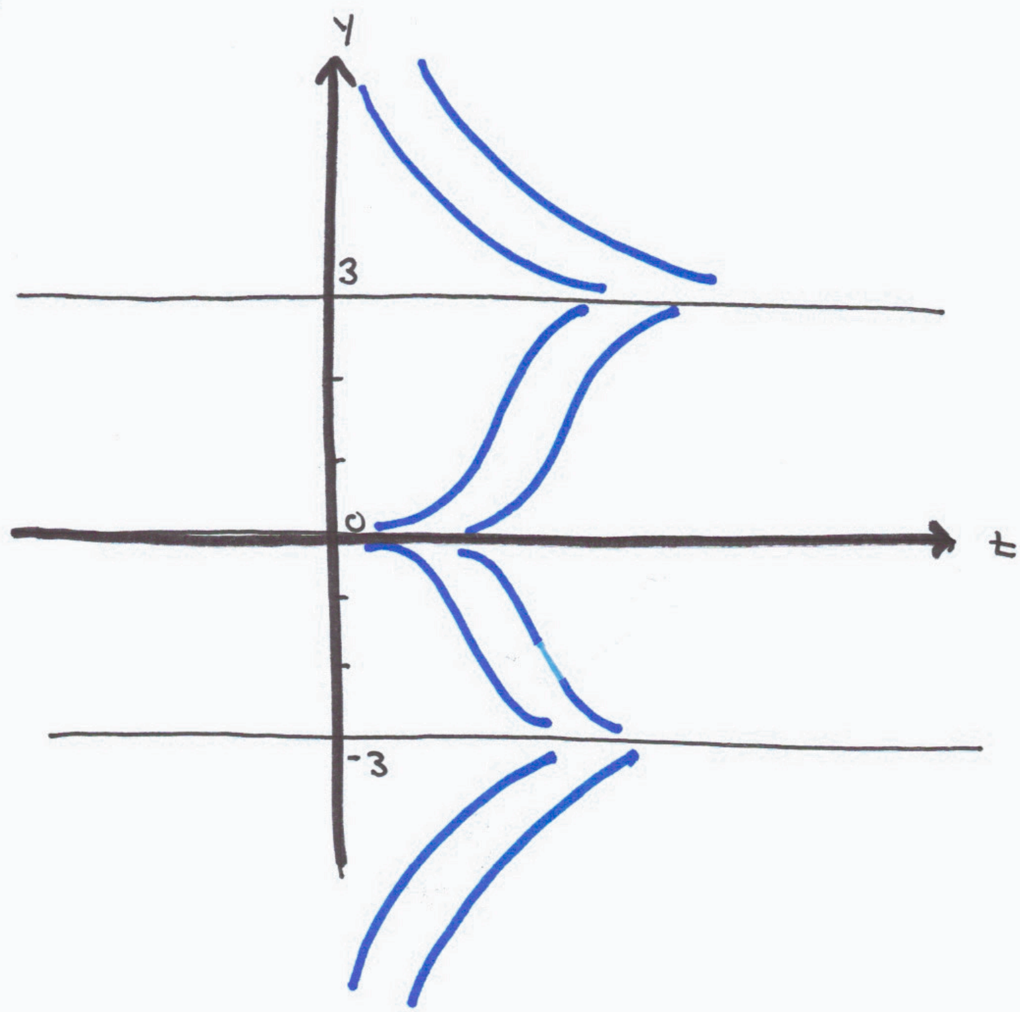
$0 = y(9 - y^2)$

$0 = y(3 - y)(3 + y)$

$y = 0,\ y = 3,\ y = -3$



b)

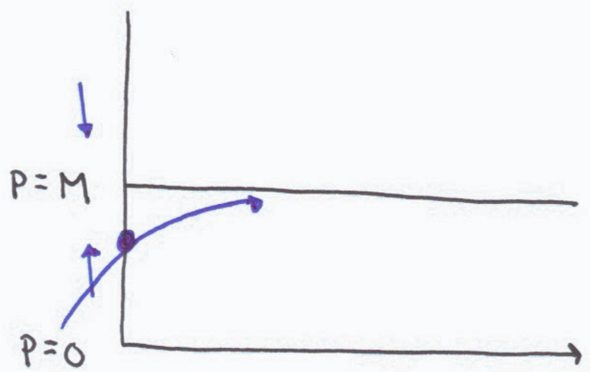


Problem 10

$$\frac{dP}{dt} = P(M-P)$$

$$0 = P(M-P)$$

$$P=0,\ P=M$$



$$\boxed{M > 50}$$

Problem 11

$$v' = -k\sqrt{v}, \qquad v(0) = 100, \; v(2) = 64$$

$$\int \frac{1}{\sqrt{v}}\,dv = \int -k\,dt$$

$$2\sqrt{v} = -kt + c$$

$$\sqrt{v} = -\frac{1}{2}kt + c$$

$$v = \left(-\frac{1}{2}kt + c\right)^2$$

$$v(0) = 100 \rightarrow \quad 100 = \left(-\frac{1}{2}k(0) + c\right)^2$$

$$c^2 = 100$$

$$c = 10$$

$$v = \left(-\frac{1}{2}kt + 10\right)^2$$

$$v(2) = 64 \rightarrow \quad 64 = \left(-\frac{1}{2}k \cdot 2 + 10\right)^2$$

$$64 = (10 - k)^2$$

$$10 - k = 8$$

$$k = 2$$

$$\boxed{v(t) = (-t + 10)^2}$$

$$x(t) = \int (-t + 10)^2\,dt = -\frac{1}{3}(-t + 10)^3 + D$$

$$x(0) = 0 \rightarrow \quad 0 = -\frac{1}{3}(10)^3 + D$$

$$D = \frac{1000}{3}$$

$$x(t) = -\frac{1}{3}(-t+10)^3 + \frac{1000}{3}$$

Boat stops when $v(t) = 0 \rightarrow t = 10$

$$\boxed{x(10) = \frac{1000}{3} \text{ ft}}$$

Problem 12

$$\frac{dx}{dt} = r_i c_i - \frac{r_o x}{V(t)}$$

$V(0) = 100$

$x(0) = 20$

$c_i = 1$

$r_i = 4$

$r_o = 3$

$$V(t) = V_0 + (r_i - r_o)t$$

$$V(t) = 100 + (4-3)t$$

$$V(t) = 100 + t$$

$$\frac{dx}{dt} = (4)(1) - \frac{3x}{100+t}$$

$$x' + \frac{3}{100+t}x = 4$$ → use integrating factor method to solve

$$\mu(t) = e^{\int \frac{3}{100+t} dt} = e^{3\ln(100+t)} = e^{\ln(100+t)^3} = (100+t)^3$$

$$x(t) = \frac{1}{(100+t)^3}\left(\int 4(100+t)^3 dt\right)$$

$$x(t) = \frac{1}{(100+t)^3}\left((100+t)^4 + c\right)$$

$$\boxed{x(t) = 100 + t + \frac{c}{(100+t)^3}}$$

Problem 13

a) $r^2 + 3 = 0$

$r^2 = -3$

$r = 0 \pm \sqrt{3}i$

$Y = C_1 e^{0t} \cos(\sqrt{3}t) + C_2 e^{0t} \sin(\sqrt{3}t)$

$\boxed{Y = C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t)}$

b) $r^2 + r - 2 = 0$

$(r + 2)(r - 1) = 0$

$r = -2, r = 1$

$\boxed{Y = C_1 e^{-2t} + C_2 e^{t}}$

c) $r^2 + 6r + 9 = 0$

$(r + 3)(r + 3) = 0$

$r = -3, r = -3$

$\boxed{Y = C_1 e^{-3t} + C_2 t e^{-3t}}$

Problem 14

$r^2 + 4r + 4 = 0$

$(r+2)(r+2) = 0$

$r = -2, r = -2 \rightarrow$ Repeated

$y = C_1 e^{-2t} + C_2 t e^{-2t}$

$y(0) = -6$

$-6 = C_1$

$y = -6e^{-2t} + C_2 t e^{-2t}$

$y' = 12e^{-2t} - 2C_2 t e^{-2t} + C_2 e^{-2t}$

$y'(0) = 8$

$8 = 12 + C_2$

$C_2 = -4$

$$\boxed{y = -6e^{-2t} - 4te^{-2t}}$$

Problem 15

$$y'' + 9y = 5\sin(2x)$$

Find y_c:

$$r^2 + 9 = 0$$
$$r^2 = -9$$
$$r = \pm 3i$$

$$y_c = C_1\cos(3x) + C_2\sin(3x)$$

Find y_p:

$$y_p = A\cos(2x) + B\sin(2x)$$
$$y_p' = -2A\sin(2x) + 2B\cos(2x)$$
$$y_p'' = -4A\cos(2x) - 4B\sin(2x)$$

$$y'' + 9y = 5\sin(2x)$$

$$-4A\cos(2x) - 4B\sin(2x) + 9A\cos(2x) + 9B\sin(2x) = 5\sin(2x)$$

$$5A\cos(2x) + 5B\sin(2x) = 0\cos(2x) + 5\sin(2x)$$

$$A = 0, \quad 5B = 5$$
$$B = 1$$

$$y_p = \sin(2x)$$

$$y = y_c + y_p$$
$$y = C_1\cos(3x) + C_2\sin(3x) + \sin(2x)$$

$$y(0) = 0 \rightarrow 0 = C_1$$

$$y = C_2\sin(3x) + \sin(2x)$$
$$y' = 3C_2\cos(3x) + 2\cos(2x)$$

$$y'(0) = 8 \rightarrow 8 = 3C_2 + 2$$
$$6 = 3C_2$$
$$C_2 = 2$$

$$y = 2\sin(3x) + \sin(2x)$$

Problem 16

$$y'' + 25y = 3\cos(5x)$$

Find y_c:

$r^2 + 25 = 0$

$r = \pm 5i$

$$y_c = C_1\cos(5x) + C_2\sin(5x)$$

Find y_p:

$y_p = A\cos(5x) + B\sin(5x)$ → need to modify

$y_p = x(A\cos(5x) + B\sin(5x))$

$y_p' = x(-5A\sin(5x) + 5B\cos(5x)) + A\cos(5x) + B\sin(5x)$

$y_p'' = x(-25A\cos(5x) - 25B\sin(5x)) - 5A\sin(5x) + 5B\cos(5x)$

$\quad - 5A\sin(5x) + 5B\cos(5x)$

$= -25Ax\cos(5x) - 25Bx\sin(5x) - 10A\sin(5x) + 10B\cos(5x)$

$y'' + 25y = 3\cos(5x)$

$-25Ax\cos(5x) - 25Bx\sin(5x) - 10A\sin(5x) + 10B\cos(5x)$

$+ 25Ax\cos(5x) + 25Bx\sin(5x) = 3\cos(5x)$

$-10A\sin(5x) + 10B\cos(5x) = 3\cos(5x)$

$A = 0, \quad B = \frac{3}{10}$

$y_p = \frac{3}{10}x\sin(5x)$

$$y = C_1\cos(5x) + C_2\sin(5x) + \frac{3}{10}x\sin(5x)$$

Problem 17

a) True

b) False $A = \sqrt{C_1^2 + C_2^2}$

$A = \sqrt{(2)^2 + (2)^2} = \sqrt{8}$

c) False. It will remain constant

Problem 18

a) $x'' + 4x' + 9x = 0$

$r^2 + 4r + 9 = 0$

$r = \dfrac{-4 \pm \sqrt{16 - 4(1)(9)}}{2} = \dfrac{-4 \pm \sqrt{-20}}{2} = \dfrac{-4 \pm 2\sqrt{5}i}{2} = -2 \pm \sqrt{5}i$

$x(t) = C_1 e^{-2t} \cos(\sqrt{5}t) + C_2 e^{-2t} \sin(\sqrt{5}t)$

$x(0) = 0 \rightarrow \quad 0 = C_1 + 0$

$C_1 = 0$

$x(t) = C_2 e^{-2t} \sin(\sqrt{5}t)$

$x'(t) = C_2 e^{-2t} \cdot \sqrt{5}\cos(\sqrt{5}t) - 2C_2 e^{-2t} \sin(\sqrt{5}t)$

$x'(0) = 1 \rightarrow \quad 1 = \sqrt{5}C_2 - 0$

$C_2 = \dfrac{1}{\sqrt{5}}$

$x(t) = \dfrac{1}{\sqrt{5}} e^{-2t} \sin(\sqrt{5}t)$

b) underdamped

c) $x(t) = 0$

$\dfrac{1}{\sqrt{5}} e^{-2t} \sin(\sqrt{5}t) = 0$

$\sin(\sqrt{5}t) = 0$

$\sqrt{5}t = 0 + 2\pi n$

$t = \dfrac{2\pi n}{\sqrt{5}}$

Problem 19

a) $x(t) = 0$

$$3e^{-4t} - 5e^{-3t} = 0$$

$$3e^{-4t} = 5e^{-3t}$$

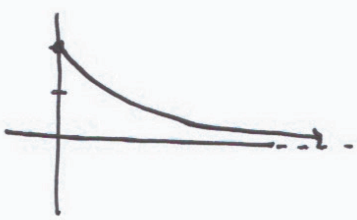
$$\frac{3}{5} = \frac{e^{-3t}}{e^{-4t}}$$

$\frac{3}{5} = e^{t} \rightarrow t = \ln\left(\frac{3}{5}\right)$, but this is a negative number so it does <u>NOT</u> pass through eq.

b+c) Initial Pos.: $x(0) = 3e^{0} - 5e^{0} = -2 \rightarrow$ above eq.

$$x'(t) = -12e^{-4t} + 15e^{-3t}$$

Initial Vel.: $x'(0) = -12 + 15 = 3 \rightarrow$ downward



most stretched: as $t \rightarrow \infty$

most compressed: at $t = 0$

Problem 20

$m = 1$

$c = 0$

$k = 9$

$x(0) = 0$

$x'(0) = 0$

a) $x'' + 9x = 10\cos(3t), \quad x(0) = 0, \; x'(0) = 0$

b) Find x_c:

$r^2 + 9 = 0$

$r = \pm 3i$

$x_c = c_1\cos(3t) + c_2\sin(3t)$

Find x_p:

$x_p = A\cos(3t) + B\sin(3t)$ → need to modify

$x_p = t(A\cos(3t) + B\sin(3t))$

$x_p' = t(-3A\sin(3t) + 3B\cos(3t)) + A\cos(3t) + B\sin(3t)$

$x_p'' = t(-9A\cos(3t) - 9B\sin(3t)) - 3A\sin(3t) + 3B\cos(3t) - 3A\sin(3t) + 3B\cos(3t)$

$= -9At\cos(3t) - 9Bt\sin(3t) - 6A\sin(3t) + 6B\cos(3t)$

$x'' + 9x = 10\cos(3t)$

$9At\cos(3t) - 9Bt\sin(3t) - 6A\sin(3t) + 6B\cos(3t) + 9At\cos(3t) + 9Bt\sin(3t) = 10\cos(3t)$

$-6A\sin(3t) + 6B\cos(3t) = 0\sin(3t) + 10\cos(3t)$

$A = 0, \quad 6B = 10$

$B = \frac{10}{6} = \frac{5}{3}$

$x_p = \frac{5}{3}t\sin(3t)$

$x(t) = x_c + x_p$

$x(t) = c_1\cos(3t) + c_2\sin(3t) + \frac{5t}{3}\sin(3t)$

$x(0) = 0 \rightarrow 0 = C_1 + 0 + 0$

$C_1 = 0$

$x(t) = C_2 \sin(3t) + \frac{5}{3}t \sin(3t)$

$x'(t) = 3C_2 \cos(3t) + \frac{5}{3}t(3\cos(3t)) + \sin(3t) \cdot \frac{5}{3}$

$x'(0) = 0 \rightarrow 0 = 3C_2 + 0 + 0$

$C_2 = 0$

$x(t) = \frac{5}{3}t \sin(3t)$

c) yes

Problem 21

Start by finding the characteristic equation:

$$(1-r)(3-r)-8=0$$
$$3-4r+r^2-8=0$$
$$r^2-4r-5=0$$
$$r=5, r=-1$$

For $r=5$, find the associated k:

$A-rI$

$$\begin{bmatrix} 1-5 & 2 \\ 4 & 3-5 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$$

So $k_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For $r=-1$, find the associated k:

$A-rI$

$$\begin{bmatrix} 1+1 & 2 \\ 4 & 3+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$$

So $k_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$x(t) = C_1 k_1 e^{r_1 t} + C_2 k_2 e^{r_2 t}$$

$$x(t) = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

Problem 22

Start by finding the characteristic equation:

$$r^2 - 8r + 16 = 0$$
$$(r-4)(r-4) = 0$$
$$r = 4, r = 4$$

For $r = 4$, find the associated k:

$A - rI$

$\begin{bmatrix} 3-4 & 1 \\ -1 & 5-4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

So $k = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and there is only 1 linearly independent eigenvector. Therefore, the solution will take the form:

$x(t) = C_1 k e^{rt} + C_2(kte^{rt} + \eta e^{rt})$

To find η:

$(A - rI)\eta = k$

$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$-1\eta_1 + 1\eta_2 = 1$

Choose any η_1 and η_2 that satisfy the above equation:

$\eta_1 = 0, \eta_2 = 1$

So $\eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and the overall general solution is:

$x(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + C_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{4t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t} \right)$

Use the initial value of $x(0) = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

$\begin{bmatrix} 4 \\ -3 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

$4 = C_1$
$-3 = C_1 + C_2$

$C_2 = -7$

$x(t) = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} - 7 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{4t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t} \right)$

$x(t) = \begin{bmatrix} 4e^{4t} - 7te^{4t} \\ -3e^{4t} - 7t3^{4t} \end{bmatrix}$

Problem 23

Start by finding the eigenvalues:

$$(2-r)^2+16=0$$
$$(2-r)^2=-16$$
$$2-r=\pm 4i$$
$$r=2\pm 4i$$

For $r=2+4i$, find the associated k:

$$\begin{bmatrix} 2-(2+4i) & 4 \\ -4 & 2-(2+4i) \end{bmatrix} = \begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix}$$

So $k_1 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} i$

$$x(t) = C_1 e^{2t}\left(\begin{bmatrix} 4 \\ 0 \end{bmatrix}\cos(4t) - \begin{bmatrix} 0 \\ 4 \end{bmatrix}\sin(4t)\right) + C_2 e^{2t}\left(\begin{bmatrix} 4 \\ 0 \end{bmatrix}\sin(4t) + \begin{bmatrix} 0 \\ 4 \end{bmatrix}\cos(4t)\right)$$

$$\begin{cases} 3 = 4C_1 \\ 8 = 4C_2 \end{cases}$$

$$C_2 = 3/4$$
$$C_1 = 2$$

$$x(t) = \frac{3}{4}e^{2t}\left(\begin{bmatrix} 4 \\ 0 \end{bmatrix}\cos(4t) - \begin{bmatrix} 0 \\ 4 \end{bmatrix}\sin(4t)\right) + 2e^{2t}\left(\begin{bmatrix} 4 \\ 0 \end{bmatrix}\sin(4t) + \begin{bmatrix} 0 \\ 4 \end{bmatrix}\cos(4t)\right)$$

$$x(t) = \begin{bmatrix} e^{2t}(3\cos(4t) + 6\sin(4t)) \\ e^{2t}(-3\sin(4t) + 6\cos(4t)) \end{bmatrix}$$

Problem 24

Start by finding the eigenvalues:

$$
\begin{gathered}
(-2-r)(4-r)+5=0\\
-8+2r-4r+r^2+5=0\\
r^2-2r-3=0\\
(r-3)(r+1)=0\\
r_1=3, r_2=-1
\end{gathered}
$$

For $r_1 = 3$, find the associated k_1:

$$\begin{bmatrix} -2-3 & 1 \\ -5 & 4-3 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -5 & 1 \end{bmatrix}$$

So $k_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

For $r_2 = -1$, find the associated k_1:

$$\begin{bmatrix} -2+1 & 1 \\ -5 & 4+1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix}$$

So $k_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$x(t) = C_1 k_1 e^{r_1 t} + C_2 k_2 e^{r_2 t}$$

$$x(t) = C_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

$$\begin{cases} 1 = C_1 + C_2 \\ 3 = 5C_1 + C_2 \end{cases}$$

$$
\begin{gathered}
3 = 5(1-C_2)+C_2\\
3 = 5-4C_2\\
C_2 = 1/2\\
C_1 = 1/2
\end{gathered}
$$

$$x(t) = \frac{1}{2}\begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{3t} + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

$$x(t) = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ \frac{5}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}$$

Problem 25

Manipulate the integral to get in the form $\int_0^\infty e^{-st} f(t)\, dt$, then take the Laplace transform of $f(t)$.

$$\int_0^\infty e^{-st} e^{-4t} \cos(7t)\, dt$$

$$f(t) = e^{-4t} \cos\,(7t)$$

$$\mathcal{L}\{e^{-4t} \cos(7t)\} = \frac{s+4}{(s+4)^2 + 49}$$

Problem 26

a. Use Partial Fractions, then take the inverse Laplace.

$$\frac{36-s}{s(s+6)^2} = \frac{A}{s} + \frac{B}{(s+6)} + \frac{C}{(s+6)^2}$$

$$36 - s = A(s+6)^2 + Bs(s+6) + Cs$$

$$s = 0 \to A = 1$$
$$s = -6 \to C = -7$$

$$36 - s = 1(s+6)^2 + Bs(s+6) - 7s$$

$$36 - s = s^2 + 12s + 36 + Bs^2 + 6Bs - 7s$$

Equating the coefficients gives us $1 + B = 0$, so $B = -1$

$$\frac{36-s}{s(s+6)^2} = \frac{1}{s} - \frac{1}{(s+6)} - \frac{7}{(s+6)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{(s+6)} - \frac{7}{(s+6)^2}\right\} = \mathbf{1 - e^{-6t} - 7e^{-6t}t}$$

b. Since the denominator does not factor, start by completing the square.

$$\frac{2s-2}{s^2 + 2s \mathbf{+1} + 17 \mathbf{-1}}$$

$$2\frac{s-1}{(s+1)^2 + 16}$$

$$2\frac{s \mathbf{+1} - 1 \mathbf{-1}}{(s+1)^2 + 16}$$

$$2\left[\frac{s+1}{(s+1)^2 + 16} - \frac{2}{(s+1)^2 + 16}\right]$$

Now take the inverse Laplace

$$2\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2 + 16} - \frac{1}{\mathbf{2}}\frac{2 * \mathbf{2}}{(s+1)^2 + 16}\right]$$

$$= 2\left[e^{-t}\cos(4t) - \frac{1}{2}e^{-t}\sin(4t)\right]$$