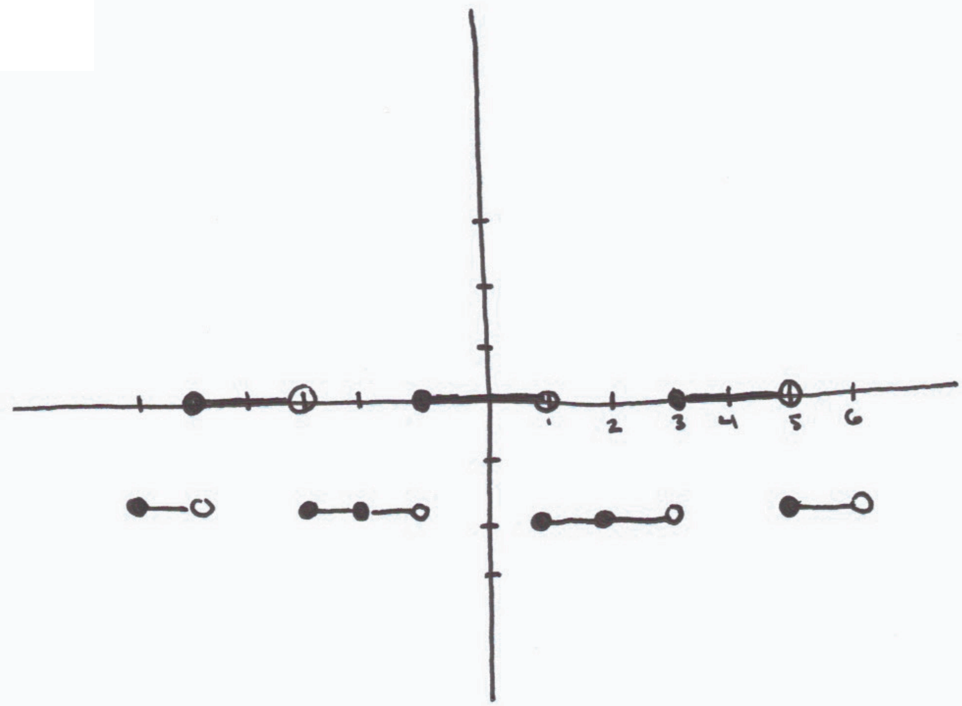
Problem 1



$$x = 1 \rightarrow \frac{\lim\limits_{x \to 1^-} f(x) + \lim\limits_{x \to 1^+} f(x)}{2} = \frac{0 - 2}{2} = \boxed{-1}$$

$$x = 2 \rightarrow \boxed{-2}$$

$$x = 3 \rightarrow \frac{\lim\limits_{x \to 3^-} f(x) + \lim\limits_{x \to 3^+} f(x)}{2} = \boxed{-1}$$

Problem 2

$L = \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\cancel{\int_{-\pi}^{0} 0\,dx}^{\,0} + \int_0^{\pi} 2x\,dx\right]$$

$$= \frac{1}{\pi}\left[x^2\Big|_0^{\pi}\right] = \frac{1}{\pi}\left[\pi^2 - 0\right] = \frac{\pi^2}{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\left[\cancel{\int_{-\pi}^{0} 0\cdot\cos\left(\frac{n\pi x}{\pi}\right)dx}^{\,0} + \int_0^{\pi} 2x\cdot\cos\left(\frac{n\pi x}{\pi}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} 2x\cdot\cos(nx)\,dx\right] \rightarrow$$ Integration By Parts

$u = 2x$ $\quad$ $v = \frac{1}{n}\sin(nx)$

$du = 2dx$ $\quad$ $dv = \cos(nx)dx$

$$= \frac{1}{\pi}\left[\frac{2x}{n}\sin(nx) - \int \frac{2}{n}\sin(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{2x}{n}\sin(nx) + \frac{2}{n^2}\cos(nx)\right]\Bigg|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\cancel{\frac{2\pi}{n}\sin(n\pi)}^{\,0} + \frac{2}{n^2}\cos(n\pi)\right) - \left(\cancel{0}^{\,0} + \frac{2}{n^2}\cos(0)\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{2}{n^2}(-1)^n - \frac{2}{n^2}(1)\right]$$

$$= \frac{1}{\pi}\left[\frac{2}{n^2}\left((-1)^n - 1\right)\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$= 0$, when n is even $(n = 2k)$

$= -2$, when n is odd $(n = 2k-1)$

$$= \begin{cases} 0 & n = 2k \\ \dfrac{-4}{\pi(2k-1)^2} & n = 2k-1 \end{cases} = \frac{-4}{\pi(2k-1)^2}$$

$$b_n = \frac{1}{\pi}\left[\cancelto{0}{\int_{-\pi}^{0} 0\cdot \sin\left(\frac{n\pi x}{\pi}\right)dx} + \int_{0}^{\pi} 2x\sin\left(\frac{n\pi x}{\pi}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\int_{0}^{\pi} 2x\sin(nx)\,dx\right] \rightarrow$$

I.B.P.

$u = 2x$ $\qquad$ $v = \frac{-1}{n}\cos(nx)$

$du = 2dx$ $\qquad$ $dv = \sin(nx)$

$$= \frac{1}{\pi}\left[-\frac{2x}{n}\cos(nx) + \int \frac{2}{n}\cos(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{2x}{n}\cos(nx) + \frac{2}{n^2}\sin(nx)\right]\Bigg|_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(-\frac{2\pi}{n}\cos(n\pi) + \cancelto{0}{\frac{2}{n^2}\sin(n\pi)}\right) - \cancelto{0}{\left(0 + \frac{2}{n^2}\sin(0)\right)}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}(-1)^n\right] = -\frac{2}{n}(-1)^n \quad \text{OR} \quad \frac{2(-1)^{n+1}}{n}$$

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right)$$

$$\boxed{F(x) = \frac{\pi}{2} + \sum_{k=1}^{\infty}\left(\frac{-4}{\pi(2k-1)^2}\cos((2k-1)\pi)\right) + \sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n}\sin(nx)\right)}$$

Problem 3

Heat Equation

$$8u_{xx} = u_t, \qquad u(0,t)=0,\ u(4,t)=0 \rightarrow \text{Case I}$$

$L=4$

$\alpha^2 = 8$

$$u(x,0) = x \rightarrow f(x)$$

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{2}\int_0^4 x\sin\left(\frac{n\pi x}{4}\right)dx \longrightarrow$$

I.B.P.

$u = x \qquad v = \frac{-4}{n\pi}\cos\left(\frac{n\pi x}{4}\right)$

$du = dx \qquad dv = \sin\left(\frac{n\pi x}{4}\right)dx$

$$= \frac{1}{2}\left[\frac{-4}{n\pi}x\cos\left(\frac{n\pi x}{4}\right) + \int \frac{4}{n\pi}\cos\left(\frac{n\pi x}{4}\right)dx\right]$$

$$= \frac{1}{2}\left[\frac{-4}{n\pi}x\cos\left(\frac{n\pi x}{4}\right) + \frac{4}{n\pi}\cdot\frac{4}{n\pi}\sin\left(\frac{n\pi x}{4}\right)\right]\Bigg|_0^4$$

$$= \frac{1}{2}\cdot\frac{4}{n\pi}\left[-x\cos\left(\frac{n\pi x}{4}\right) + \frac{4}{n\pi}\sin\left(\frac{n\pi x}{4}\right)\right]\Bigg|_0^4$$

$$= \frac{2}{n\pi}\left[\left(-4\cos(n\pi) + \cancel{\frac{4}{n\pi}\sin(n\pi)}^{\,0}\right) - (0+0)\right]$$

$$= \frac{2}{n\pi}\left(-4(-1)^n\right) = \frac{8(-1)^{n+1}}{n\pi}$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{\frac{-\alpha^2 n^2\pi^2 t}{L^2}}\sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{u(x,t) = \sum \frac{8(-1)^{n+1}}{n\pi}\, e^{-\frac{8n^2\pi^2 t}{16}}\sin\left(\frac{n\pi x}{4}\right)}$$

Problem 4

Heat Equation

$L = 8$

$5u_{xx} = u_t \qquad u_x(0,t) = 0, \; u_x(8,t) = 0 \rightarrow$ case 2

$\alpha^2 = 5$

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n e^{-\frac{25n^2\pi^2 t}{64}} \cos\left(\frac{n\pi x}{8}\right)$$

$$u(x,0) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{8}\right)$$ $\rightarrow$ set equal to $u(x,0)$ given in problem to solve for C_n

$$7 - 3\cos\left(\frac{\pi x}{4}\right) + 12\cos\left(\frac{\pi x}{2}\right) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{8}\right)$$

$n=2$, $n=4$

$C_0 = 7 \qquad C_2 = -3 \qquad C_4 = 12$, all other $C_n = 0$.

$$u(x,t) = 7 + C_2 e^{-\frac{25(2)^2\pi^2 t}{64}} \cos\left(\frac{2\pi x}{8}\right) + C_4 e^{-\frac{25(4)^2\pi^2 t}{64}} \cos\left(\frac{4\pi x}{8}\right)$$

$$u(x,t) = 7 - 3e^{-\frac{25}{16}\pi^2 t} \cos\left(\frac{\pi x}{4}\right) + 12e^{-\frac{25}{4}\pi^2 t} \cos\left(\frac{\pi x}{2}\right)$$

Problem 5

Wave Equation

$9u_{xx} = u_{tt}$ $\qquad u(0,t)=0, \; u(3,t)=0 \rightarrow L=3$

$a^2 = 9$
$a = 3$

$u(x,0) = x \qquad u_t(x,0) = 0$

$\downarrow f(x) \qquad \downarrow g(x)$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{3}\int_0^3 x\sin\left(\frac{n\pi x}{3}\right)dx \longrightarrow \text{I.B.P.} \quad u = x,\; du = dx \quad v = -\frac{3}{n\pi}\cos\left(\frac{n\pi x}{3}\right),\; dv = \sin\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[-\frac{3}{n\pi}x\cos\left(\frac{n\pi x}{3}\right) + \int \frac{3}{n\pi}\cos\left(\frac{n\pi x}{3}\right)dx\right]$$

$$= \frac{2}{3}\cdot\frac{3}{n\pi}\left[-x\cos\left(\frac{n\pi x}{3}\right) + \frac{3}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\right]\Bigg|_0^3$$

$$= \frac{2}{n\pi}\left[\left(-3\cos(n\pi) + \cancel{\frac{3}{n\pi}\sin(n\pi)}^{\,0}\right) - (0)\right]$$

$$= -\frac{6}{n\pi}(-1)^n = \frac{6}{n\pi}(-1)^{n+1}$$

$B_n = 0$

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{6}{n\pi}(-1)^{n+1}\cos\left(\frac{\cancel{3}n\pi t}{\cancel{3}}\right)\sin\left(\frac{n\pi x}{3}\right)\right]$$

Problem 6

Wave Equation

$u_{xx} = u_{tt}$ $\qquad u(0,t)=0, \quad u(8,t)=0$ $\qquad L = 8$

$a = 1$

$$u(x,t) = \sum_{n=1}^{\infty} \left[\left(A_n \cos\left(\frac{n\pi t}{8}\right) + B_n \sin\left(\frac{n\pi t}{8}\right) \right) \sin\left(\frac{n\pi x}{8}\right) \right]$$

$$u(x,0) = \sum_{n=1}^{\infty} \left[A_n \sin\left(\frac{n\pi x}{8}\right) \right]$$

$$\frac{1}{2}\sin\left(\underbrace{\frac{\pi x}{8}}_{n=1}\right) - 3\sin\left(\underbrace{\frac{\pi x}{4}}_{n=2}\right) = A_n \sin\left(\frac{n\pi x}{8}\right)$$

$A_1 = \frac{1}{2}$ $\qquad A_2 = -3$ $\qquad$ all other $A_n = 0$

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[-\frac{n\pi}{8} A_n \sin\left(\frac{n\pi t}{8}\right) + \frac{n\pi}{8} B_n \cos\left(\frac{n\pi t}{8}\right) \right] \sin\left(\frac{n\pi x}{8}\right)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \left[\frac{n\pi}{8} B_n \sin\left(\frac{n\pi x}{8}\right) \right]$$

$$5\sin\left(\underbrace{\frac{\pi x}{2}}_{n=4}\right) = \frac{n\pi}{8} B_n \sin\left(\frac{n\pi x}{8}\right)$$

$5 = \frac{4\pi}{8} B_4 \rightarrow B_4 = \frac{10}{\pi}$ $\qquad$ all other $B_n = 0$

$$u(x,t) = \frac{1}{2}\cos\left(\frac{1\pi t}{8}\right)\sin\left(\frac{\pi x}{8}\right) - 3\cos\left(\frac{2\pi t}{8}\right)\sin\left(\frac{2\pi x}{8}\right) + \frac{10}{\pi}\sin\left(\frac{4\pi t}{8}\right)\sin\left(\frac{4\pi x}{8}\right)$$

Problem 7

$$1 + u(t-1)(t^2-1) + u(t-2\pi)(\cos t - t^2)$$

$$\mathcal{L}\{1\} + \mathcal{L}\{u(t-1)(t^2-1)\} + \mathcal{L}\{u(t-2\pi)(\cos t - t^2)\}$$

$$\frac{1}{s} + e^{-s}\,\mathcal{L}\{(t+1)^2 - 1\} + e^{-2\pi s}\mathcal{L}\{\cos(t+2\pi) - (t+2\pi)^2\}$$

$$\frac{1}{s} + e^{-s}\,\mathcal{L}\{t^2 + 2t + 1 - 1\} + e^{-2\pi s}\mathcal{L}\{\cos(t) - (t^2 + 4\pi t + 4\pi^2)\}$$

$$\frac{1}{s} + e^{-s}\,\mathcal{L}\{t^2 + 2t\} + e^{-2\pi s}\mathcal{L}\{\cos(t) - t^2 - 4\pi t - 4\pi^2\}$$

$$\frac{1}{s} + e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2}\right) + e^{-2\pi s}\left(\frac{s}{s^2+1} - \frac{2}{s^3} - \frac{4\pi}{s^2} - \frac{4\pi^2}{s}\right)$$

Problem 8

Take inverse Laplace of each term separately. You will have to complete the square for the first term…

$$\mathcal{L}^{-1}\left\{e^{-3s}\frac{s}{s^2+6s+\mathbf{9}+25-\mathbf{9}}-5e^{-4s}\right\}$$

$$\mathcal{L}^{-1}\left\{e^{-3s}\frac{s}{(s+3)^2+16}-5e^{-4s}\right\}$$

$$\mathcal{L}^{-1}\left\{e^{-3s}\frac{s+\mathbf{3}-\mathbf{3}}{(s+3)^2+16}-5e^{-4s}\right\}$$

$$\mathcal{L}^{-1}\left\{e^{-3s}\left(\frac{s+3}{(s+3)^2+16}-\frac{3}{(s+3)^2+16}\right)-5e^{-4s}\right\}$$

$$\mathcal{L}^{-1}\left\{e^{-3s}\left(\frac{s+3}{(s+3)^2+16}-\frac{3}{(s+3)^2+16}\right)\right\}-\mathcal{L}^{-1}\{5e^{-4s}\}$$

$$=u(t-3)\mathcal{L}^{-1}\left\{\left(\frac{s+3}{(s+3)^2+16}-\frac{3}{(s+3)^2+16}\right)\bigg|_{t=t-3}\right\}-5\delta(t-4)$$

$$=u(t-3)\mathcal{L}^{-1}\left\{\left(\frac{s+3}{(s+3)^2+16}-\frac{\mathbf{1}}{\mathbf{4}}\frac{3\mathbf{(4)}}{(s+3)^2+16}\right)\bigg|_{t=t-3}\right\}-5\delta(t-4)$$

$$=u(t-3)\left[e^{-3(t-3)}\cos\big(4(t-3)\big)-\frac{3}{4}e^{-3(t-3)}\sin\big(4(t-3)\big)\right]-5\delta(t-4)$$

Problem 9

Start by finding the Laplace transform of both sides.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = \mathcal{L}\{2\delta(t-1)\}$$

$$s^2Y - s \cdot y(0) - y'(0) + 4[sY - y(0)] + 3Y = 2e^{-s}$$

$$s^2Y - 2 + 4sY + 3Y = 2e^{-s}$$

$$s^2Y + 4sY + 3Y = 2e^{-s} + 2$$

$$(s^2 + 4s + 3)Y = 2e^{-s} + 2$$

$$Y = e^{-s}\frac{2}{(s+3)(s+1)} + \frac{2}{(s+3)(s+1)}$$

$$y = \mathcal{L}^{-1}\left\{e^{-s}\frac{2}{(s+3)(s+1)} + \frac{2}{(s+3)(s+1)}\right\}$$

Partial Fraction Decomposition:

$$\frac{2}{(s+3)(s+1)} = \frac{A}{s+3} + \frac{B}{s+1} \qquad \rightarrow A = -1, B = 1$$

Back to the equation:

$$y = \mathcal{L}^{-1}\left\{e^{-s}\left[\frac{-1}{s+3} + \frac{1}{s+1}\right]\right\} + \mathcal{L}^{-1}\left\{\frac{-1}{s+3} + \frac{1}{s+1}\right\}$$

$$y = u(t-1)\mathcal{L}^{-1}\left\{\frac{-1}{s+3} + \frac{1}{s+1}\bigg|_{t=t-1}\right\} + \mathcal{L}^{-1}\left\{\frac{-1}{s+3} + \frac{1}{s+1}\right\}$$

$$y = u(t-1)\left(-e^{-3(t-1)} + e^{-(t-1)}\right) - e^{-3t} + e^{-t}$$

Problem 10

Start by finding the Laplace transform of both sides.

$$\mathcal{L}\{y''\} - 9\mathcal{L}\{y\} = \mathcal{L}\{u(t-6)\} + \mathcal{L}\{\delta(t-3)\}$$

$$s^2Y - s \cdot y(0) - y'(0) - 9Y = \frac{e^{-6s}}{s} + e^{-3s}$$

$$s^2Y - 9Y = \frac{e^{-6s}}{s} + e^{-3s}$$

$$Y(s^2 - 9) = \frac{e^{-6s}}{s} + e^{-3s}$$

$$Y = \frac{e^{-6s}}{s(s+3)(s-3)} + \frac{e^{-3s}}{(s+3)(s-3)}$$

$$y = \mathcal{L}^{-1}\left\{e^{-6s}\frac{1}{s(s+3)(s-3)}\right\} + \mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{(s+3)(s-3)}\right\}$$

PFD #1:

$$\frac{1}{s(s+3)(s-3)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-3}$$

$$A = -\frac{1}{9}, B = \frac{1}{18}, C = \frac{1}{18}$$

PFD #2:

$$\frac{1}{(s+3)(s-3)} = \frac{A}{s+3} + \frac{B}{s-3}$$

$$A = -\frac{1}{6}, B = \frac{1}{6}$$

$$y = \mathcal{L}^{-1}\left\{e^{-6s}\left(-\frac{1}{9}\cdot\frac{1}{s} + \frac{1}{18}\cdot\frac{1}{s+3} + \frac{1}{18}\cdot\frac{1}{s-3}\right)\right\} + \mathcal{L}^{-1}\left\{e^{-3s}\left(-\frac{1}{6}\cdot\frac{1}{s+3} + \frac{1}{6}\cdot\frac{1}{s-3}\right)\right\}$$

$$y = u(t-6)\,\mathcal{L}^{-1}\left(-\frac{1}{9}\cdot\frac{1}{s} + \frac{1}{18}\cdot\frac{1}{s+3} + \frac{1}{18}\cdot\frac{1}{s-3}\right)\Bigg|_{t=t-6} + u(t-3)\mathcal{L}^{-1}\left(-\frac{1}{6}\cdot\frac{1}{s+3} + \frac{1}{6}\cdot\frac{1}{s-3}\right)\Bigg|_{t=t-3}$$

$$y = u(t-6)\left(-\frac{1}{9} + \frac{1}{18}e^{-3(t-6)} + \frac{1}{18}e^{3(t-6)}\right) + u(t-3)\left(-\frac{1}{6}e^{-3(t-3)} + \frac{1}{6}e^{3(t-3)}\right)$$