

Problem 1

a) order = 2

b) linear

$$\left. \begin{array}{l} y = c \\ y' = 0 \\ y'' = 0 \end{array} \right\} \rightarrow \begin{array}{l} 3c - 0 + 0 = 1 \\ c = \frac{1}{3} \end{array}$$

$$\boxed{y = \frac{1}{3}}$$

Problem 2

a) True

b) True

c) False  $\rightarrow$  If  $y=0, y'=0$

$$\begin{array}{l} yy' = 10 \\ (0)(0) = 10 \end{array}$$

$$0 = 10 \leftarrow \text{NOT TRUE}$$

d) True  $\rightarrow$  If  $y=1, y'=0$

$$0 - t^2 \sin(\pi) = 0$$

$$0 = 0 \leftarrow \text{TRUE}$$

Problem 3

$$y = t^m$$

$$y' = m t^{m-1}$$

$$y'' = m(m-1)t^{m-2}$$

$$t^2 y'' - 6y = 0$$

$$t^2 (m(m-1)t^{m-2}) - 6t^m = 0$$

$$m(m-1)t^m - 6t^m = 0$$

$$t^m (m(m-1) - 6) = 0$$

$$t^m (m^2 - m - 6) = 0$$

$$~~t^m~~ (m-3)(m+2) = 0$$

$$m = 3, m = -2$$

**B**

Problem 4

a)  $\frac{dT}{dt} = k(A-T)$

$A = 27$

$\frac{dT}{dt} = k(27-T)$

$T(0) = 3$

$T(4) = 15$

$\int \frac{1}{27-T} dT = \int k dt$

$-\ln(27-T) = kt + C$

$\ln(27-T) = -kt + C$

$27-T = ce^{-kt}$

$T = 27 - ce^{-kt}$

$\downarrow T(0)=3 \rightarrow 3 = 27 - ce^0$   
 $C = 24$

$T = 27 - 24e^{-kt}$

$T(4)=15 \rightarrow 15 = 27 - 24e^{-4k}$

$-12 = -24e^{-4k}$

$\frac{1}{2} = e^{-4k}$

$\ln(\frac{1}{2}) = -4k$

$k = \frac{\ln(\frac{1}{2})}{-4} = \frac{-\ln(2)}{-4} = \frac{\ln(2)}{4}$

$T(t) = 27 - 24e^{\frac{-\ln(2)}{4}t}$

b)  $19 = 27 - 24e^{\frac{-\ln(2)}{4}t}$

$\frac{1}{3} = e^{\frac{\ln(2)}{4}t}$

$\ln(\frac{1}{3}) = \frac{-\ln(2)}{4}t$

$\rightarrow t = \frac{4\ln(\frac{1}{3})}{-\ln(2)} = \frac{-4\ln(3)}{-\ln(2)} =$

$\frac{4\ln(3)}{\ln(2)} \text{ min.}$

Use the Integrating Factor Method.

$$y' + \frac{1}{2\sqrt{t}} y = 1$$

$$p(t) = \frac{1}{2\sqrt{t}}$$

$$g(t) = 1$$

$$\mu(t) = e^{\int p(t) dt} = e^{\int \frac{1}{2} t^{-1/2} dt} = e^{\sqrt{t}}$$

$$y = \frac{1}{e^{\sqrt{t}}} \left[ \int e^{\sqrt{t}} \cdot 1 dt \right]$$

To evaluate  $\int e^{\sqrt{t}} dt$  use substitution and I.B.P.

$$\int e^{\sqrt{t}} dt \rightarrow \begin{cases} w = \sqrt{t} \\ dw = \frac{1}{2\sqrt{t}} dt \\ dt = 2\sqrt{t} dw \end{cases}$$

$$= \int e^w 2\sqrt{t} dw$$

$$= \int 2we^w dw \rightarrow \begin{cases} u = 2w \\ du = 2 dw \end{cases} \quad \begin{matrix} v = e^w \\ dv = e^w dw \end{matrix}$$

$$= 2we^w - \int 2e^w dw$$

$$= 2we^w - 2e^w + c$$

$$= 2\sqrt{t}e^{\sqrt{t}} - 2e^{\sqrt{t}} + c$$

$$y = \frac{1}{e^{\sqrt{t}}} \left[ 2\sqrt{t}e^{\sqrt{t}} - 2e^{\sqrt{t}} + c \right]$$

$$y = 2\sqrt{t} - 2 + \frac{c}{e\sqrt{t}}$$

$$y'' = -2 \rightarrow -2 = 2 - 2 + \frac{c}{e}$$

$$-2 = \frac{c}{e}$$

$$c = -2e$$

$$y = 2\sqrt{t} - 2 - \frac{2e}{e\sqrt{t}}$$

Problem 6

$$y' + \underbrace{2t}_{p} y = \underbrace{e^{-t^2} \cos t + t^3}_q$$

$$u(t) = e^{\int 2t dt} = e^{t^2}$$

$$y = \frac{1}{e^{t^2}} \left[ \int e^{t^2} (e^{-t^2} \cos t + t^3) dt \right]$$

$$y = \frac{1}{e^{t^2}} \left[ \int \cos t dt + \int t^3 e^{t^2} dt \right]$$

$$\int \cos(t) dt = \sin t$$

$$\int t^3 e^{t^2} dt$$

$$\int t^3 e^w \frac{dw}{2t}$$

$$\int \frac{1}{2} t^2 e^w dw$$

$$\frac{1}{2} \int w e^w \rightarrow$$

$$\frac{1}{2} [w e^w - \int e^w dw]$$

$$\frac{1}{2} [t^2 e^{t^2} - e^{t^2}] + c$$

Substitution

$$w = t^2 \\ dw = 2t dt \\ dt = \frac{dw}{2t}$$

I.B.P.

$$u = w \\ du = dw, \quad v = e^w \\ dv = e^w dw$$

$$y = \frac{1}{e^{t^2}} \left[ \sin t + \frac{1}{2} t^2 e^{t^2} - \frac{1}{2} e^{t^2} + c \right]$$

$$y = \frac{\sin t}{e^{t^2}} + \frac{1}{2} t^2 - \frac{1}{2} + \frac{c}{e^{t^2}}$$

Problem 7

This is a separable ODE...

$$\frac{dy}{dx} = \frac{2\cos(2x)}{y+1}$$

$$\int (y+1) dy = \int 2\cos(2x) dx$$

$$\frac{1}{2}y^2 + y = \sin(2x) + C$$

$$y^2 + 2y = 2\sin(2x) + C$$

$$y(0) = 3 \rightarrow (3)^2 + 2(3) = 2\sin(0) + C$$
$$15 = C$$

$$y^2 + 2y + 1 = 2\sin(2x) + 15 + 1$$

$$(y+1)^2 = 2\sin(2x) + 16$$

$$y+1 = \pm \sqrt{2\sin(2x) + 16}$$

$$y = -1 \pm \sqrt{2\sin(2x) + 16}, \text{ but since } y(0) = 3 \text{ we only want the positive root}$$

$$y = -1 + \sqrt{2\sin(2x) + 16}$$

Problem 8

$$\begin{aligned}
 \text{a) } \left. \begin{array}{l} y = k \\ y' = 0 \end{array} \right\} &\rightarrow \begin{array}{l} 0 = k^2 - 2k \\ 0 = k(k-2) \\ k = 0, k = 2 \end{array}
 \end{aligned}$$

constant sol.  $y = 0, y = 2$

$$\text{b) } \frac{dy}{dt} = y^2 - 2y$$

$$\frac{1}{y^2 - 2y} dy = 1 dt$$

$$\int \frac{1}{y(y-2)} dy = \int 1 dt$$

P.F.D.

$$\frac{1}{y(y-2)} = \frac{A}{y} + \frac{B}{y-2}$$

$$A = -\frac{1}{2}, B = \frac{1}{2}$$

$$\int \left( \frac{1/2}{y-2} - \frac{1/2}{y} \right) dy = \int 1 dt$$

$$\frac{1}{2} [\ln|y-2| - \ln|y|] = t + C$$

$$\ln \sqrt{\frac{y-2}{y}} = t + C$$

$$\sqrt{\frac{y-2}{y}} = e^{t+C}$$

$$\sqrt{\frac{y-2}{y}} = ce^t$$

$$\frac{y-2}{y} = (ce^t)^2$$

$$\frac{y-2}{y} = ce^{2t}$$

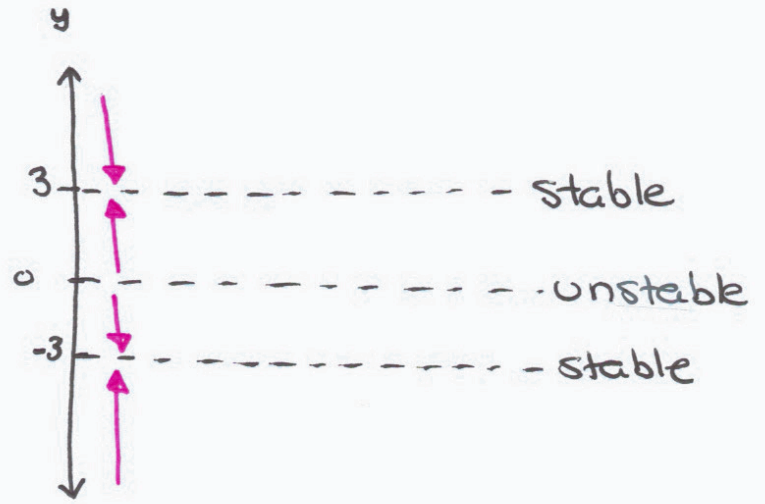
$$\begin{aligned}
 y-2 &= yce^{2t} \\
 y - yce^{2t} &= 2 \\
 y(1 - ce^{2t}) &= 2
 \end{aligned}$$

$$y = \frac{2}{1 - ce^{2t}}$$

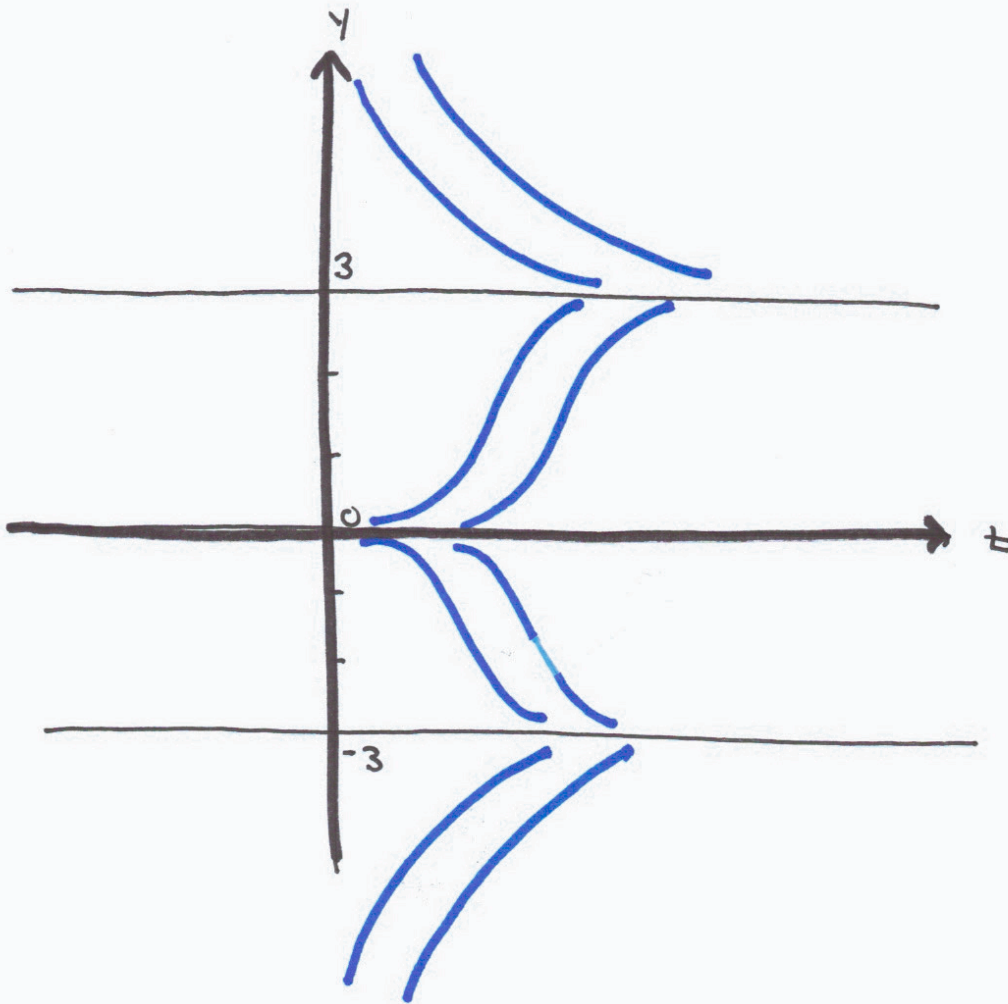


Problem 9

a)  $0 = 9y - y^3$   
 $0 = y(9 - y^2)$   
 $0 = y(3 - y)(3 + y)$   
 $y = 0, y = 3, y = -3$



b)

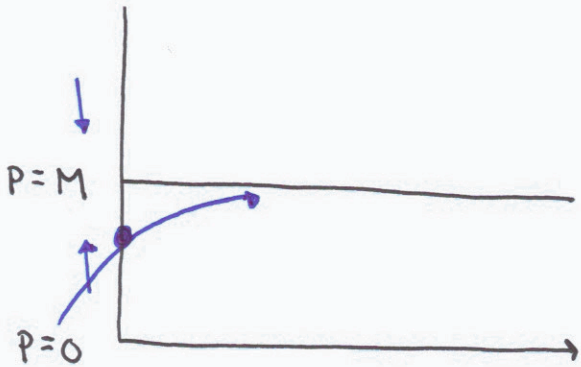


Problem 10

$$\frac{dP}{dt} = P(M-P)$$

$$0 = P(M-P)$$

$$P=0, P=M$$



$$M > 50$$

Problem 11

$$v' = -k\sqrt{v}, \quad v(0) = 100, \quad v(2) = 64$$

$$\int \frac{1}{\sqrt{v}} dv = \int -k dt$$

$$2\sqrt{v} = -kt + C$$

$$\sqrt{v} = -\frac{1}{2}kt + C$$

$$v = \left(-\frac{1}{2}kt + C\right)^2$$

$$v(0) = 100 \rightarrow 100 = \left(-\frac{1}{2}k(0) + C\right)^2$$
$$C^2 = 100$$
$$C = 10$$

$$v = \left(-\frac{1}{2}kt + 10\right)^2$$

$$v(2) = 64 \rightarrow 64 = \left(-\frac{1}{2}k \cdot 2 + 10\right)^2$$

$$64 = (10 - k)^2$$

$$10 - k = 8$$

$$k = 2$$

$$\boxed{v(t) = (-t + 10)^2}$$

$$x(t) = \int (-t + 10)^2 dt = -\frac{1}{3}(-t + 10)^3 + D$$

$$x(0) = 0 \rightarrow 0 = -\frac{1}{3}(10)^3 + D$$

$$D = \frac{1000}{3}$$

$$x(t) = -\frac{1}{3}(-t + 10)^3 + \frac{1000}{3}$$

Boat stops when  $v(t) = 0 \rightarrow t = 10$

$$x(10) = \frac{1000}{3} \text{ ft}$$

Problem 12

$$\frac{dx}{dt} = r_i C_i - \frac{r_o X}{V(t)}$$

$$V(0) = 100$$

$$X(0) = 20$$

$$C_i = 1$$

$$r_i = 4$$

$$r_o = 3$$

$$\rightarrow V(t) = V_0 + (r_i - r_o)t$$

$$V(t) = 100 + (4 - 3)t$$

$$V(t) = 100 + t$$

$$\frac{dx}{dt} = (4)(1) - \frac{3X}{100+t}$$

$$X' + \frac{3}{100+t}X = 4 \rightarrow \text{use integrating factor method to solve}$$

$$\mu(t) = e^{\int \frac{3}{100+t} dt} = e^{3 \ln(100+t)} = e^{\ln(100+t)^3} = (100+t)^3$$

$$x(t) = \frac{1}{(100+t)^3} \left( \int 4(100+t)^3 dt \right)$$

$$x(t) = \frac{1}{(100+t)^3} \left( (100+t)^4 + C \right)$$

$$x(t) = 100 + t + \frac{C}{(100+t)^3}$$

Problem 13

$$a) r^2 + 3 = 0$$

$$r^2 = -3$$

$$r = 0 \pm \sqrt{3}i$$

$$y = c_1 e^{0t} \cos(\sqrt{3}t) + c_2 e^{0t} \sin(\sqrt{3}t)$$

$$y = c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}t)$$

$$b) r^2 + r - 2 = 0$$

$$(r + 2)(r - 1) = 0$$

$$r = -2, r = 1$$

$$y = c_1 e^{-2t} + c_2 e^t$$

$$c) r^2 + 6r + 9 = 0$$

$$(r + 3)(r + 3) = 0$$

$$r = -3, r = -3$$

$$y = c_1 e^{-3t} + c_2 t e^{-3t}$$

Problem 14

$$r^2 + 4r + 4 = 0$$

$$(r+2)(r+2) = 0$$

$$r = -2, r = -2 \rightarrow \text{Repeated}$$

$$y = C_1 e^{-2t} + C_2 t e^{-2t}$$

$$y(0) = -6$$

$$-6 = C_1$$

$$y = -6e^{-2t} + C_2 t e^{-2t}$$

$$y' = 12e^{-2t} - 2C_2 t e^{-2t} + C_2 e^{-2t}$$

$$y'(0) = 8$$

$$8 = 12 + C_2$$

$$C_2 = -4$$

$$y = -6e^{-2t} - 4t e^{-2t}$$

Problem 15

$$y'' + 9y = 5 \sin(2x)$$

Find  $y_c$ :

$$r^2 + 9 = 0$$

$$r^2 = -9$$

$$r = \pm 3i$$

$$y_c = C_1 \cos(3x) + C_2 \sin(3x)$$

Find  $y_p$ :

$$y_p = A \cos(2x) + B \sin(2x)$$

$$y_p' = -2A \sin(2x) + 2B \cos(2x)$$

$$y_p'' = -4A \cos(2x) - 4B \sin(2x)$$

$$y'' + 9y = 5 \sin(2x)$$

$$-4A \cos(2x) - 4B \sin(2x) + 9A \cos(2x) + 9B \sin(2x) = 5 \sin(2x)$$

$$5A \cos(2x) + 5B \sin(2x) = 0 \cos(2x) + 5 \sin(2x)$$

$$A = 0, \quad \begin{matrix} 5B = 5 \\ B = 1 \end{matrix}$$

$$y_p = \sin(2x)$$

$$y = y_c + y_p$$

$$y = C_1 \cos(3x) + C_2 \sin(3x) + \sin(2x)$$

$$y(0) = 0 \rightarrow 0 = C_1$$

$$y = C_2 \sin(3x) + \sin(2x)$$

$$y' = 3C_2 \cos(3x) + 2 \cos(2x)$$

$$y'(0) = 8 \rightarrow \begin{matrix} 8 = 3C_2 + 2 \\ 6 = 3C_2 \\ C_2 = 2 \end{matrix}$$

$$y = 2 \sin(3x) + \sin(2x)$$



Problem 16

$$y'' + 25y = 3\cos(5x)$$

Find  $y_c$ :

$$r^2 + 25 = 0$$
$$r = \pm 5i$$

$$y_c = C_1 \cos(5x) + C_2 \sin(5x)$$

Find  $y_p$ :

$$y_p = A \cos(5x) + B \sin(5x) \rightarrow \text{need to modify}$$

$$y_p = x(A \cos(5x) + B \sin(5x))$$

$$y_p' = x(-5A \sin(5x) + 5B \cos(5x)) + A \cos(5x) + B \sin(5x)$$

$$y_p'' = x(-25A \cos(5x) - 25B \sin(5x)) - 5A \sin(5x) + 5B \cos(5x) - 5A \sin(5x) + 5B \cos(5x)$$

$$= -25Ax \cos(5x) - 25Bx \sin(5x) - 10A \sin(5x) + 10B \cos(5x)$$

$$y'' + 25y = 3\cos(5x)$$

$$-25Ax \cos(5x) - 25Bx \sin(5x) - 10A \sin(5x) + 10B \cos(5x)$$

$$+ 25Ax \cos(5x) + 25Bx \sin(5x) = 3\cos(5x)$$

$$-10A \sin(5x) + 10B \cos(5x) = 3\cos(5x)$$

$$A = 0, \quad B = \frac{3}{10}$$

$$y_p = \frac{3}{10} x \sin(5x)$$

$$y = C_1 \cos(5x) + C_2 \sin(5x) + \frac{3}{10} x \sin(5x)$$

Problem 17

a) True

b) False

$$A = \sqrt{C_1^2 + C_2^2}$$

$$A = \sqrt{(2)^2 + (2)^2} = \sqrt{8}$$

c) False. It will remain constant

Problem 18

$$a) x'' + 4x' + 9x = 0$$

$$r^2 + 4r + 9 = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 4(1)(9)}}{2} = \frac{-4 \pm \sqrt{-20}}{2} = \frac{-4 \pm 2\sqrt{5}i}{2} = -2 \pm \sqrt{5}i$$

$$x(t) = C_1 e^{-2t} \cos(\sqrt{5}t) + C_2 e^{-2t} \sin(\sqrt{5}t)$$

$$x(0) = 0 \rightarrow 0 = C_1 + 0$$

$$C_1 = 0$$

$$x(t) = C_2 e^{-2t} \sin(\sqrt{5}t)$$

$$x'(t) = C_2 e^{-2t} \cdot \sqrt{5} \cos(\sqrt{5}t) - 2C_2 e^{-2t} \sin(\sqrt{5}t)$$

$$x'(0) = 1 \rightarrow 1 = \sqrt{5} C_2 - 0$$

$$C_2 = \frac{1}{\sqrt{5}}$$

$$x(t) = \frac{1}{\sqrt{5}} e^{-2t} \sin(\sqrt{5}t)$$

b) underdamped

$$c) x(t) = 0$$

$$\frac{1}{\sqrt{5}} e^{-2t} \sin(\sqrt{5}t) = 0$$

$$\sin(\sqrt{5}t) = 0$$

$$\sqrt{5}t = 0 + 2\pi n$$

$$t = \frac{2\pi n}{\sqrt{5}}$$

Problem 19

a)  $x(t) = 0$

$$3e^{-4t} - 5e^{-3t} = 0$$

$$3e^{-4t} = 5e^{-3t}$$

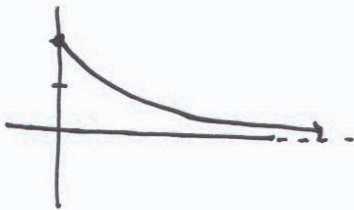
$$\frac{3}{5} = \frac{e^{-3t}}{e^{-4t}}$$

$\frac{3}{5} = e^t \rightarrow t = \ln\left(\frac{3}{5}\right)$ , but this is a negative number so it does NOT pass through eq.

b+c) Initial Pos.:  $x(0) = 3e^0 - 5e^0 = -2 \rightarrow$  above eq.

$$x'(t) = -12e^{-4t} + 15e^{-3t}$$

Initial Vel.:  $x'(0) = -12 + 15 = 3 \rightarrow$  downward



most stretched: as  $t \rightarrow \infty$

most compressed: at  $t = 0$

Problem 20

$$m = 1$$

$$c = 0$$

$$k = 9$$

$$x(0) = 0$$

$$x'(0) = 0$$

$$a) \quad x'' + 9x = 10\cos(3t), \quad x(0) = 0, \quad x'(0) = 0$$

b) Find  $x_c$ :

$$r^2 + 9 = 0$$

$$r = \pm 3i$$

$$x_c = c_1 \cos(3t) + c_2 \sin(3t)$$

Find  $x_p$ :

$$x_p = A \cos(3t) + B \sin(3t) \rightarrow \text{need to modify}$$

$$x_p = t(A \cos(3t) + B \sin(3t))$$

$$x_p' = t(-3A \sin(3t) + 3B \cos(3t)) + A \cos(3t) + B \sin(3t)$$

$$x_p'' = t(-9A \cos(3t) - 9B \sin(3t)) - 3A \sin(3t) + 3B \cos(3t) - 3A \sin(3t) + 3B \cos(3t) \\ = -9At \cos(3t) - 9Bt \sin(3t) - 6A \sin(3t) + 6B \cos(3t)$$

$$x'' + 9x = 10\cos(3t)$$

$$9At \cos(3t) - 9Bt \sin(3t) - 6A \sin(3t) + 6B \cos(3t) + 9At \cos(3t) + 9Bt \sin(3t) = 10\cos(3t)$$

$$-6A \sin(3t) + 6B \cos(3t) = 0 \sin(3t) + 10 \cos(3t)$$

$$A = 0,$$

$$6B = 10$$

$$B = \frac{10}{6} = \frac{5}{3}$$

$$x_p = \frac{5}{3} t \sin(3t)$$

$$x(t) = x_c + x_p$$

$$x(t) = c_1 \cos(3t) + c_2 \sin(3t) + \frac{5}{3} t \sin(3t)$$

$$x(0) = 0 \rightarrow 0 = C_1 + 0 + 0$$

$$C_1 = 0$$

$$x(t) = C_2 \sin(3t) + \frac{5t}{3} \sin(3t)$$

$$x'(t) = 3C_2 \cos(3t) + \frac{5t}{3} (3 \cos(3t)) + \sin(3t) \cdot \frac{5}{3}$$

$$x'(0) = 0 \rightarrow 0 = 3C_2 + 0 + 0$$

$$C_2 = 0$$

$$x(t) = \frac{5t}{3} \sin(3t)$$

c) yes