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## MATH 250 Final Exam (New Material) – Sample Test – Detailed Solutions

1.

Start by finding the characteristic equation:

$$(1 - r)(3 - r) - 8 = 0$$

$$3 - 4r + r^2 - 8 = 0$$

$$r^2 - 4r - 5 = 0$$

$$r = 5, r = -1$$

For  $r = 5$ , find the associated  $k$ :

$$A - rI$$

$$\begin{bmatrix} 1 - 5 & 2 \\ 4 & 3 - 5 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$$

$$\text{So } k_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For  $r = -1$ , find the associated  $k$ :

$$A - rI$$

$$\begin{bmatrix} 1 + 1 & 2 \\ 4 & 3 + 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$$

$$\text{So } k_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$x(t) = C_1 k_1 e^{r_1 t} + C_2 k_2 e^{r_2 t}$$

$$x(t) = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

2.

Start by finding the characteristic equation:

$$\begin{aligned}r^2 - 8r + 16 &= 0 \\(r - 4)(r - 4) &= 0 \\r &= 4, r = 4\end{aligned}$$

For  $r = 4$ , find the associated  $k$ :

$$A - rI$$

$$\begin{bmatrix} 3 - 4 & 1 \\ -1 & 5 - 4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

So  $k = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , and there is only 1 linearly independent eigenvector. Therefore, the solution will take the form:

$$x(t) = C_1 k e^{rt} + C_2 (k t e^{rt} + \eta e^{rt})$$

To find  $\eta$ :

$$(A - rI)\eta = k$$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$-1\eta_1 + 1\eta_2 = 1$$

Choose any  $\eta_1$  and  $\eta_2$  that satisfy the above equation:

$$\eta_1 = 0, \eta_2 = 1$$

So  $\eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , and the overall general solution is:

$$x(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + C_2 \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{4t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t} \right)$$

Use the initial value of  $x(0) = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

$$\begin{bmatrix} 4 \\ -3 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$4 = C_1$$

$$-3 = C_1 + C_2$$

$$C_2 = -7$$

$$x(t) = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} - 7 \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{4t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t} \right)$$

$$x(t) = \begin{bmatrix} 4e^{4t} - 7te^{4t} \\ -3e^{4t} - 7t3^{4t} \end{bmatrix}$$

3.

Start by finding the eigenvalues:

$$\begin{aligned}(2-r)^2 + 16 &= 0 \\ (2-r)^2 &= -16 \\ 2-r &= \pm 4i \\ r &= 2 \pm 4i\end{aligned}$$

For  $r = 2 + 4i$ , find the associated  $k$ :

$$\begin{bmatrix} 2 - (2 + 4i) & 4 \\ -4 & 2 - (2 + 4i) \end{bmatrix} = \begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix}$$

$$\text{So } k_1 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} i$$

$$x(t) = C_1 e^{2t} \left( \begin{bmatrix} 4 \\ 0 \end{bmatrix} \cos(4t) - \begin{bmatrix} 0 \\ 4 \end{bmatrix} \sin(4t) \right) + C_2 e^{2t} \left( \begin{bmatrix} 4 \\ 0 \end{bmatrix} \sin(4t) + \begin{bmatrix} 0 \\ 4 \end{bmatrix} \cos(4t) \right)$$

$$\begin{cases} 3 = 4C_1 \\ 8 = 4C_2 \end{cases}$$

$$\begin{aligned} C_2 &= 3/4 \\ C_1 &= 2 \end{aligned}$$

$$x(t) = \frac{3}{4} e^{2t} \left( \begin{bmatrix} 4 \\ 0 \end{bmatrix} \cos(4t) - \begin{bmatrix} 0 \\ 4 \end{bmatrix} \sin(4t) \right) + 2e^{2t} \left( \begin{bmatrix} 4 \\ 0 \end{bmatrix} \sin(4t) + \begin{bmatrix} 0 \\ 4 \end{bmatrix} \cos(4t) \right)$$

$$x(t) = \begin{bmatrix} e^{2t}(3 \cos(4t) + 6 \sin(4t)) \\ e^{2t}(-3 \sin(4t) + 6 \cos(4t)) \end{bmatrix}$$

4.

Start by finding the eigenvalues:

$$\begin{aligned}(-2 - r)(4 - r) + 5 &= 0 \\ -8 + 2r - 4r + r^2 + 5 &= 0 \\ r^2 - 2r - 3 &= 0 \\ (r - 3)(r + 1) &= 0 \\ r_1 = 3, r_2 &= -1\end{aligned}$$

For  $r_1 = 3$ , find the associated  $k_1$ :

$$\begin{bmatrix} -2 - 3 & 1 \\ -5 & 4 - 3 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -5 & 1 \end{bmatrix}$$

$$\text{So } k_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

For  $r_2 = -1$ , find the associated  $k_1$ :

$$\begin{bmatrix} -2 + 1 & 1 \\ -5 & 4 + 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix}$$

$$\text{So } k_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(t) = C_1 k_1 e^{r_1 t} + C_2 k_2 e^{r_2 t}$$

$$x(t) = C_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

$$\begin{cases} 1 = C_1 + C_2 \\ 3 = 5C_1 + C_2 \end{cases}$$

$$3 = 5(1 - C_2) + C_2$$

$$3 = 5 - 4C_2$$

$$C_2 = 1/2$$

$$C_1 = 1/2$$

$$x(t) = \frac{1}{2} \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{3t} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

$$x(t) = \begin{bmatrix} \frac{1}{2} e^{3t} + \frac{1}{2} e^{-t} \\ \frac{5}{2} e^{3t} + \frac{1}{2} e^{-t} \end{bmatrix}$$

5. **Linearly independent since  $W \neq 0$**

Solution:

$$W(x_1, x_2) = \begin{bmatrix} 2e^{5t} \cos t & 4e^{5t} \sin t \\ 4e^{5t} \sin t & -8e^{5t} \cos t \end{bmatrix}$$

$$W(x_1, x_2) = -16e^{10t} \cos^2 t - 16e^{10t} \sin^2 t = -16e^{10t}(\sin^2 t + \cos^2 t) = -16e^{10t} \neq 0.$$

6. **Linearly Dependent since  $W = 0$**

$$W(x_1, x_2) = \begin{bmatrix} 3e^{-t} & -6e^{-t} \\ -2e^t & 4e^{-t} \end{bmatrix} = 12e^{-2t} - 12e^{-2t} = 0$$

7.

Manipulate the integral to get in the form  $\int_0^{\infty} e^{-st} f(t) dt$ , then take the Laplace transform of  $f(t)$ .

$$\int_0^{\infty} e^{-st} e^{-4t} \cos(7t) dt$$

$$f(t) = e^{-4t} \cos(7t)$$

$$\mathcal{L}\{e^{-4t} \cos(7t)\} = \frac{s + 4}{(s + 4)^2 + 49}$$

8.

$$1 + u(t - 1)(t^2 - 1) + u(t - 2\pi)(\cos t - t^2)$$

$$\mathcal{L}\{1\} + \mathcal{L}\{u(t - 1)(t^2 - 1)\} + \mathcal{L}\{u(t - 2\pi)(\cos t - t^2)\}$$

$$\frac{1}{s} + e^{-s} \mathcal{L}\{(t + 1)^2 - 1\} + e^{-2\pi s} \mathcal{L}\{\cos(t + 2\pi) - (t + 2\pi)^2\}$$

$$\frac{1}{s} + e^{-s} \mathcal{L}\{t^2 + 2t + 1 - 1\} + e^{-2\pi s} \mathcal{L}\{\cos(t) - (t^2 + 4\pi t + 4\pi^2)\}$$

$$\frac{1}{s} + e^{-s} \mathcal{L}\{t^2 + 2t\} + e^{-2\pi s} \mathcal{L}\{\cos(t) - t^2 - 4\pi t - 4\pi^2\}$$

$$\frac{1}{s} + e^{-s} \left( \frac{2}{s^3} + \frac{2}{s^2} \right) + e^{-2\pi s} \left( \frac{s}{s^2 + 1} - \frac{2}{s^3} - \frac{4\pi}{s^2} - \frac{4\pi^2}{s} \right)$$

9.

a. Use Partial Fractions, then take the inverse Laplace.

$$\frac{36 - s}{s(s + 6)^2} = \frac{A}{s} + \frac{B}{s + 6} + \frac{C}{(s + 6)^2}$$

$$36 - s = A(s + 6)^2 + Bs(s + 6) + Cs$$

$$s = 0 \rightarrow A = 1$$

$$s = -6 \rightarrow C = -7$$

$$36 - s = 1(s + 6)^2 + Bs(s + 6) - 7s$$

$$36 - s = s^2 + 12s + 36 + Bs^2 + 6Bs - 7s$$

Equating the coefficients gives us  $1 + B = 0$ , so  $B = -1$

$$\frac{36 - s}{s(s + 6)^2} = \frac{1}{s} - \frac{1}{s + 6} - \frac{7}{(s + 6)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s + 6} - \frac{7}{(s + 6)^2}\right\} = \mathbf{1 - e^{-6t} - 7e^{-6t}t}$$

b. Since the denominator does not factor, start by completing the square.

$$\frac{2s - 2}{s^2 + 2s + \mathbf{1} + 17 - \mathbf{1}}$$

$$2 \frac{s - 1}{(s + 1)^2 + 16}$$

$$2 \frac{s + \mathbf{1} - 1 - \mathbf{1}}{(s + 1)^2 + 16}$$

$$2 \left[ \frac{s + 1}{(s + 1)^2 + 16} - \frac{2}{(s + 1)^2 + 16} \right]$$

Now take the inverse Laplace

$$2\mathcal{L}^{-1}\left[\frac{s + 1}{(s + 1)^2 + 16} - \frac{1}{\mathbf{2} \cdot \mathbf{2} (s + 1)^2 + 16}\right]$$

$$= 2 \left[ e^{-t} \cos(4t) - \frac{1}{2} e^{-t} \sin(4t) \right]$$

- c. Take inverse Laplace of each term separately. You will have to complete the square for the first term...

$$\mathcal{L}^{-1}\left\{e^{-3s}\frac{s}{s^2 + 6s + 9 + 25 - 9} - 5e^{-4s}\right\}$$

$$\mathcal{L}^{-1}\left\{e^{-3s}\frac{s}{(s+3)^2 + 16} - 5e^{-4s}\right\}$$

$$\mathcal{L}^{-1}\left\{e^{-3s}\frac{s+3-3}{(s+3)^2 + 16} - 5e^{-4s}\right\}$$

$$\mathcal{L}^{-1}\left\{e^{-3s}\left(\frac{s+3}{(s+3)^2 + 16} - \frac{3}{(s+3)^2 + 16}\right) - 5e^{-4s}\right\}$$

$$\mathcal{L}^{-1}\left\{e^{-3s}\left(\frac{s+3}{(s+3)^2 + 16} - \frac{3}{(s+3)^2 + 16}\right)\right\} - \mathcal{L}^{-1}\{5e^{-4s}\}$$

$$= u(t-3)\mathcal{L}^{-1}\left\{\left(\frac{s+3}{(s+3)^2 + 16} - \frac{3}{(s+3)^2 + 16}\right)\Big|_{t=t-3}\right\} - 5\delta(t-4)$$

$$= u(t-3)\mathcal{L}^{-1}\left\{\left(\frac{s+3}{(s+3)^2 + 16} - \frac{1}{4}\frac{3(4)}{(s+3)^2 + 16}\right)\Big|_{t=t-3}\right\} - 5\delta(t-4)$$

$$= u(t-3)\left[e^{-3(t-3)}\cos(4(t-3)) - \frac{3}{4}e^{-3(t-3)}\sin(4(t-3))\right] - 5\delta(t-4)$$



10. Start by finding the Laplace transform of both sides.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = \mathcal{L}\{2\delta(t-1)\}$$

$$s^2Y - s \cdot y(0) - y'(0) + 4[sY - y(0)] + 3Y = 2e^{-s}$$

$$s^2Y - 2 + 4sY + 3Y = 2e^{-s}$$

$$s^2Y + 4sY + 3Y = 2e^{-s} + 2$$

$$(s^2 + 4s + 3)Y = 2e^{-s} + 2$$

$$Y = e^{-s} \frac{2}{(s+3)(s+1)} + \frac{2}{(s+3)(s+1)}$$

$$y = \mathcal{L}^{-1} \left\{ e^{-s} \frac{2}{(s+3)(s+1)} + \frac{2}{(s+3)(s+1)} \right\}$$

Partial Fraction Decomposition:

$$\frac{2}{(s+3)(s+1)} = \frac{A}{s+3} + \frac{B}{s+1} \quad \rightarrow A = -1, B = 1$$

Back to the equation:

$$y = \mathcal{L}^{-1} \left\{ e^{-s} \left[ \frac{-1}{s+3} + \frac{1}{s+1} \right] \right\} + \mathcal{L}^{-1} \left\{ \frac{-1}{s+3} + \frac{1}{s+1} \right\}$$

$$y = u(t-1) \mathcal{L}^{-1} \left\{ \frac{-1}{s+3} + \frac{1}{s+1} \Big|_{t=t-1} \right\} + \mathcal{L}^{-1} \left\{ \frac{-1}{s+3} + \frac{1}{s+1} \right\}$$

$$y = u(t-1) (-e^{-3(t-1)} + e^{-(t-1)}) - e^{-3t} + e^{-t}$$

11. Start by finding the Laplace transform of both sides.

$$\mathcal{L}\{y''\} - 9\mathcal{L}\{y\} = \mathcal{L}\{u(t-6)\} + \mathcal{L}\{\delta(t-3)\}$$

$$s^2Y - s \cdot y(0) - y'(0) - 9Y = \frac{e^{-6s}}{s} + e^{-3s}$$

$$s^2Y - 9Y = \frac{e^{-6s}}{s} + e^{-3s}$$

$$Y(s^2 - 9) = \frac{e^{-6s}}{s} + e^{-3s}$$

$$Y = \frac{e^{-6s}}{s(s+3)(s-3)} + \frac{e^{-3s}}{(s+3)(s-3)}$$

$$y = \mathcal{L}^{-1}\left\{e^{-6s} \frac{1}{s(s+3)(s-3)}\right\} + \mathcal{L}^{-1}\left\{e^{-3s} \frac{1}{(s+3)(s-3)}\right\}$$

**PFD #1:**

$$\frac{1}{s(s+3)(s-3)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-3}$$

$$A = -\frac{1}{9}, B = \frac{1}{18}, C = \frac{1}{18}$$

**PFD #2:**

$$\frac{1}{(s+3)(s-3)} = \frac{A}{s+3} + \frac{B}{s-3}$$

$$A = -\frac{1}{6}, B = \frac{1}{6}$$

$$y = \mathcal{L}^{-1}\left\{e^{-6s} \left(-\frac{1}{9} \cdot \frac{1}{s} + \frac{1}{18} \cdot \frac{1}{s+3} + \frac{1}{18} \cdot \frac{1}{s-3}\right)\right\} + \mathcal{L}^{-1}\left\{e^{-3s} \left(-\frac{1}{6} \cdot \frac{1}{s+3} + \frac{1}{6} \cdot \frac{1}{s-3}\right)\right\}$$

$$y = u(t-6) \mathcal{L}^{-1}\left(-\frac{1}{9} \cdot \frac{1}{s} + \frac{1}{18} \cdot \frac{1}{s+3} + \frac{1}{18} \cdot \frac{1}{s-3}\right)\Big|_{t=t-6} + u(t-3) \mathcal{L}^{-1}\left(-\frac{1}{6} \cdot \frac{1}{s+3} + \frac{1}{6} \cdot \frac{1}{s-3}\right)\Big|_{t=t-3}$$

$$y = u(t-6) \left(-\frac{1}{9} + \frac{1}{18} e^{-3(t-6)} + \frac{1}{18} e^{3(t-6)}\right) + u(t-3) \left(-\frac{1}{6} e^{-3(t-3)} + \frac{1}{6} e^{3(t-3)}\right)$$