MATH 231 Exam 2 – Sample Test– Solutions

Problem 1

Use the second equation to make a polar parameterization. This gives you:

$$x = 2\cos t$$

$$z = 2\sin t$$

Then use the first equation to solve for y, and plug in the values of x and z above:

$$y = 1 + 2z - x$$

$$y = 1 + 4\sin t - 2\cos t$$

So the overall vector function is:

$$\mathbf{r}(t) = \langle 2\cos t\,, 1 + 4\sin t - 2\cos t\,, 2\sin t\rangle$$

Problem 2

a) Since one variable, y, is already solved in terms of another variable, z, we will use a trivial parameterization by letting $z = t$.

   This gives us:

$$x = 5 - t - 2t^2$$
$$y = 2t^2$$
$$z = t$$

   So $\mathbf{r}(t) = \langle 5 - t - t^2,\ 2t^2,\ t\rangle$

b) To find the equation of any line we need a point and direction vector.

   Point: $(-5,\ 8,\ 2)$

   Direction Vector: Find this by taking $\mathbf{r}'(t)$:

$$\mathbf{r}'(t) = \langle -1 - 4t, 4t, 1\rangle$$

   We want to evaluate the direction vector at $(-1, 8, 2)$ which corresponds to $t = 2$.

$$\mathbf{r}'(2) = \langle -1 - 4(2), 4(2), 1\rangle = \langle -9, 8, 1\rangle$$

   The equation of the line is:

$$\mathbf{r}(t) = \langle -5 - 9t, 8 + 8t, 2 + t\rangle$$

Problem 3

To calculate arc length first find $\mathbf{r}'(t)$.

$$r'(t) = \langle 5, -2\sin(2t), 2\cos(2t)\rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{25 + 4\sin^2(2t) + 4\cos^2(2t)} = \sqrt{29}$$

$$\int_0^3 \sqrt{29}\,dt = \sqrt{29}t = 3\sqrt{29}$$

Problem 4

$$\mathbf{v}(t) = \int \langle -\cos t, e^{2t}, -5\rangle\,dt$$

$$\mathbf{v}(t) = \langle -\sin t, \frac{1}{2}e^{2t}, -5t\rangle + C$$

To find C, use $\mathbf{v}(0) = \langle 0,3,1\rangle$

$$\langle 0,3,1\rangle = \langle -\sin 0, \frac{1}{2}e^0, -5(0)\rangle + C$$

$$\langle 0,3,1\rangle = \langle 0, \frac{1}{2}, 0\rangle + C$$

$$C = \langle 0, \frac{5}{2}, 1\rangle$$

$$\mathbf{v}(t) = \langle -\sin t, \frac{1}{2}e^{2t} + \frac{5}{2}, -5t + 1\rangle$$

$$\mathbf{r}(t) = \int \langle -\sin t, \frac{1}{2}e^{2t} + \frac{5}{2}, -5t + 1\rangle\,dt$$

$$\mathbf{r}(t) = \langle \cos t, \frac{1}{4}e^{2t} + \frac{5}{2}t, -\frac{5}{2}t^2 + t\rangle + C$$

To find C, use $\mathbf{r}(0) = \langle -1,3,2\rangle$

$$\langle -1,3,2\rangle = \langle \cos 0, \frac{1}{4}e^0 + \frac{5}{2}(0), -\frac{5}{2}(0) + (0)\rangle + C$$

$$\langle -1,3,2\rangle = \langle 1, \frac{1}{4}, 0\rangle + C$$

$$C = \langle -2, \frac{11}{4}, 2\rangle$$

$$\mathbf{r}(t) = \langle \cos t - 2, \frac{1}{4}e^{2t} + \frac{5}{2}t + \frac{11}{4}, -\frac{5}{2}t^2 + t + 2\rangle$$

Problem 5

a)

$$\mathbf{r}(t) = \int (2\mathbf{i} + 2e^{2t}\mathbf{j} + 4t\mathbf{k})\,dt$$

$$\mathbf{r}(t) = 2t\mathbf{i} + e^{2t}\mathbf{j} + 2t^2\mathbf{k} + C$$

Use $\mathbf{r}(0) = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$**,** to plug in and solve for C…

$$\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} = 2(0)\mathbf{i} + e^{2(0)}\mathbf{j} + 2(0)^2\mathbf{k} + C$$

$$\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} = \mathbf{j} + C$$

$$\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} = C$$

$$\mathbf{r}(t) = (2t+1)\mathbf{i} + (e^{2t}+2)\mathbf{j} + (2t^2+2)\mathbf{k}$$

b) First, find when speed is $\sqrt{20+4e^4}$ by setting the magnitude of the velocity equation to that value.

$$\|\mathbf{v}(t)\| = \sqrt{20+4e^4}$$

$$\sqrt{(2)^2 + (2e^{2t})^2 + (4t)^2} = \sqrt{20+4e^4}$$

$$4 + 4e^{4t} + 16t^2 = 20 + 4e^4$$

$$20t^2 + 4e^{4t} = 20 + 4e^4$$

$$t = 1$$

So this means, we want to find the acceleration at $t = 1$

$$\mathbf{v}(t) = 2\mathbf{i} + 2e^{2t}\mathbf{j} + 4t\mathbf{k}$$

$$\mathbf{a}(t) = 0\mathbf{i} + 4e^{2t}\mathbf{j} + 4\mathbf{k}$$

Problem 6

$h(90, 7) = 15$

When the plant is given 90 mL of water and 7 mg of fertilizer, its height is 15 inches.

$h_f(90, 7) = -0.5$

When the plant is given 90 mL of water and 7 mg of fertilizer, its height is decreasing at a rate of 0.5 inches/mg of fertilizer

$h_w(90, 7) = 1.2$

When the plant is given 90 mL of water and 7 mg of fertilizer, its height is increasing at a rate of 1.2 inches/mL of water

Problem 7

a) To calculate arc length first find $\mathbf{r}'(t)$.

$$\mathbf{r}'(t) = \langle 2t, t^2, 2 \rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{(2t)^2 + (t^2)^2 + (2)^2} = \sqrt{t^4 + 4t^2 + 4} = \sqrt{(t^2 + 2)^2} = t^2 + 2$$

$$\int_0^3 (t^2 + 2)\, dt = \frac{1}{3}t^3 + 2t\Big|_0^3 = 15$$

b) Point: (9, 9, 6)
Direction Vector: $\mathbf{r}'(3) = \langle 6, 9, 2 \rangle$

$$\langle 9 + 6t, 9 + 9t, 6 + 2t \rangle$$

Problem 8

a) C, because the curves are closer together
b) Negative, because as the y values increase, the value of the function (the level curves) decreases
c) Positive, because as the x values increase, the value of the function increases
d) Zero, because as the y values increase around the point, the value of the function stays the same