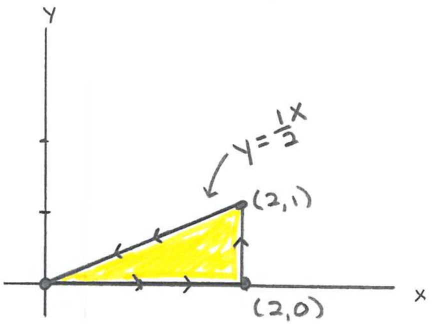
Problem 1

$$\int_C (x^2 + y^2)\,dx + (x^2 - y^2)\,dy$$

P Q



simple, closed curve so use Greene's Thm.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D (2x - 2y)\, dA$$

$$= \int_{x=0}^{x=2} \int_{y=0}^{y=\frac{1}{2}x} (2x - 2y)\, dy\, dx$$

$$= \int_{x=0}^{x=2} 2xy - y^2 \Big]_{y=0}^{y=\frac{1}{2}x} dx$$

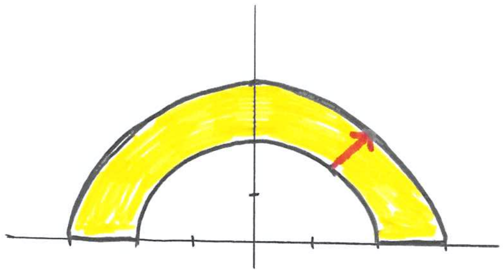
$$2x\left(\tfrac{1}{2}x\right) - \left(\tfrac{1}{2}x\right)^2 = x^2 - \tfrac{1}{4}x^2 = \tfrac{3}{4}x^2$$

$$= \int_{x=0}^{x=2} \frac{3}{4}x^2\, dx = \frac{1}{4}x^3 \Big]_0^2 = \boxed{2}$$

Problem 2

$$\oint_C (1-y^3)\,dx + (x^3+e^{y^2})\,dy$$

$(1-y^3) \downarrow P$, $(x^3+e^{y^2}) \downarrow Q$



→ simple, closed curve so use Greene's Thm

$2 \le r \le 3$

$0 \le \theta \le \pi$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D (3x^2 + 3y^2)\, dA \qquad \to 3(x^2+y^2) = 3r^2$$

$$= \int_{\theta=0}^{\theta=\pi} \int_{r=2}^{r=3} 3r^3 \, dr\, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \left. \frac{3}{4} r^4 \right]_{r=2}^{r=3} d\theta \qquad \to \frac{243}{4} - \frac{48}{4} = \frac{195}{4}$$

$$= \int_{\theta=0}^{\theta=\pi} \frac{195}{4}\, d\theta = \left. \frac{195}{4}\theta \right]_0^{\pi} = \boxed{\frac{195\pi}{4}}$$

Problem 3

Stokes Thm.

$$G(x,y) = \langle x, y, y^2 - x^2 \rangle \longrightarrow z = y^2 - x^2$$

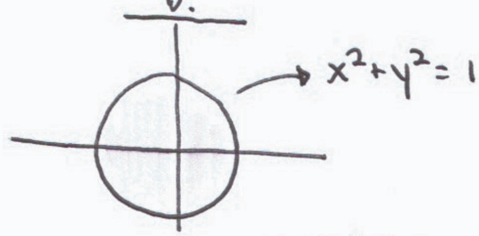
$$G_x = \langle 1, 0, -2x \rangle$$
$$G_y = \langle 0, 1, 2y \rangle$$
$$\vec{N} = \vec{G}_x \times \vec{G}_y = \langle 2x, -2y, 1 \rangle$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ x^2 y & \frac{1}{3}x^3 & xy \end{vmatrix} = \langle x, -y, 0 \rangle$$

$$\int_C \vec{F} \cdot dr = \iint_S (\text{curl } \vec{F}) \cdot dS = \iint_D \text{curl } \vec{F}(G(x,y)) \cdot \vec{N}(x,y)\, dA$$

$$= \iint_D \langle x, -y, 0 \rangle \cdot \langle 2x, -2y, 1 \rangle \, dA$$

D:



$$= \iint_D (2x^2 + 2y^2)\, dA$$

↳ D is circular so convert to polar

$0 \le r \le 1$
$0 \le \theta \le 2\pi$

$$= \iint_D 2r^2 \cdot r \, dr\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} 1 \cdot d\theta \int_{r=0}^{r=1} 2r^3 \, dr = (2\pi)\left(\frac{1}{2}\right) = \boxed{\pi}$$

Problem 4

Use Stoke's Thm

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ z^2 & y^2 & x \end{vmatrix} = \hat{i}(0) - \hat{j}(1-2z) + \hat{k}(0)$$

$$= \langle 0, 2z-1, 0 \rangle$$

To find $G(x,y)$, find the equation of the plane b/w the points: $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$

point: $(1,0,0)$

$\vec{n}$: $\langle 1,1,1 \rangle$

↳ Found by taking the cross product of 2 vectors between points on the plane

$$1(x-1) + 1(y-0) + 1(z-0) = 0$$

$$x + y + z = 1 \rightarrow z = 1 - x - y$$

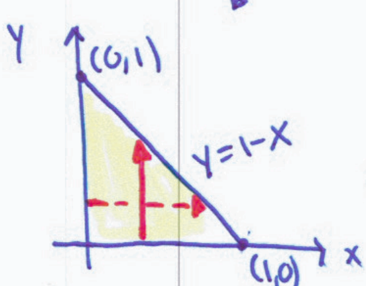
$$G(x,y) = \langle x, y, 1-x-y \rangle$$

$$G_x = \langle 1, 0, -1 \rangle, \quad G_y = \langle 0, 1, -1 \rangle$$

$$\vec{N} = G_x \times G_y = \langle 1, 1, 1 \rangle$$

$$\int_C \vec{F} \cdot dr = \iint_D \overline{(\text{curl } \vec{F}) \cdot \vec{N}} \, dA$$

$$\langle 0, 2z-1, 0 \rangle \cdot \langle 1,1,1 \rangle$$
$$= 2z - 1$$
$$= 2(1-x-y) - 1$$
$$= 2 - 2x - 2y - 1$$
$$= 1 - 2x - 2y$$



$0 \leq y \leq 1-x$

$0 \leq x \leq 1$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} (1-2x-2y)\, dy\, dx$$

$$= \int_{x=0}^{x=1} \left(y - 2xy - y^2\right)\Big|_{y=0}^{y=1-x} dx$$

$$1-x-2x(1-x) - (1-x)^2 - 0$$

$$1-x-2x+2x^2-(1-2x+x^2)$$

$$1-3x+2x^2-1+2x-x^2$$

$$x^2-x$$

$$= \int_{x=0}^{x=1} (x^2-x)\, dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 \Big|_{x=0}^{x=1} = \frac{1}{3} - \frac{1}{2} = \boxed{-\frac{1}{6}}$$

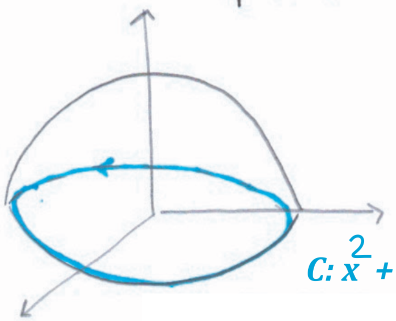
Problem 5

Special Case of Stoke's Thm

$$\iint_S \nabla \times \vec{F}\, ds$$

this is curl $\vec{F}$, so we can find another surface S_2 with the same boundary as S

$S: x^2+y^2+z^2=1$



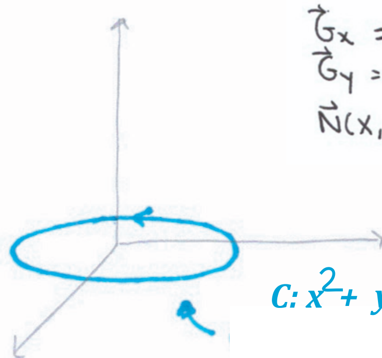
C: $x^2 + y^2 = 1$

$S_2: z=0 \rightarrow \vec{G}(x,y) = \langle x, y, 0\rangle$

$\vec{G}_x = \langle 1,0,0\rangle$

$\vec{G}_y = \langle 0,1,0\rangle$

$\vec{N}(x,y) = \vec{G}_x \times \vec{G}_y = \langle 0,0,1\rangle$



C: $x^2 + y^2 = 1$

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x^2 & e^y + x^2 & yz^4 + 2x \end{vmatrix} = \hat{i}(\text{—}) - \hat{j}(\text{—}) + \hat{k}(2-0)$$

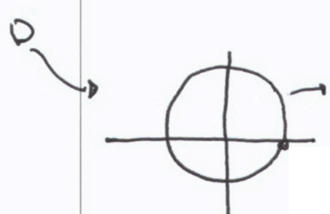
$$= \langle \text{—}, \text{—}, 2\rangle$$

$$\iint_S \text{curl}\vec{F}\cdot ds = \iint_{S_2} \text{curl}\vec{F}\cdot ds = \iint_D \text{curl}\vec{F}(G(x,y))\cdot \vec{N}(x,y)\, dA$$

$$= \iint_D \langle \text{—}, \text{—}, 2\rangle \cdot \langle 0,0,1\rangle\, dA$$

$$= \iint_D 2\, dA = 2(\text{area of } D) = 2\pi$$

D



C: $x^2 + y^2 = 1$

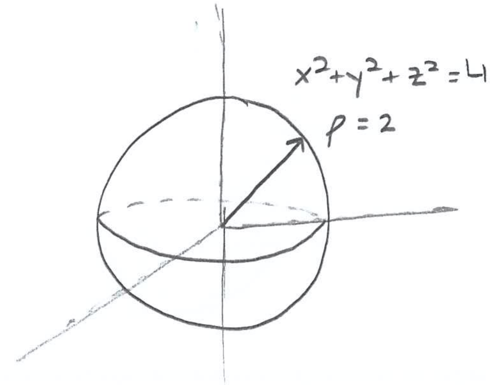
Problem 6

$$\vec{F} = \langle x + \ln(1+z),\ 2y + \sin x,\ 3z + ye^{y} \rangle$$

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 1 + 2 + 3 = 6$$

$$\iint_S \vec{F} \cdot ds = \iiint_E \operatorname{div} \vec{F} \cdot dv$$

$$= \iiint_E 6\, dv \longrightarrow \text{use spherical coordinates}$$



$0 \le \rho \le 2$
$0 \le \phi \le \pi$
$0 \le \theta \le 2\pi$

$$= \int_{\phi=0}^{\phi=\pi} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=2} 6 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_{\phi=0}^{\phi=\pi} \int_{\theta=0}^{\theta=2\pi} 2\rho^3 \sin\phi \Big]_{\rho=0}^{\rho=2} d\theta \, d\phi \longrightarrow 16 \sin\phi - 0$$

$$= \int_{\phi=0}^{\phi=\pi} \int_{\theta=0}^{\theta=2\pi} 16 \sin\phi \, d\theta \, d\phi$$

$$= \int_{\phi=0}^{\phi=\pi} 16 \sin\phi \cdot \theta \Big]_{\theta=0}^{\theta=2\pi} d\phi \longrightarrow 16 \sin\phi \cdot 2\pi - 0$$

$32\pi \sin\phi$

$$= \int_{\phi=0}^{\phi=\pi} 32\pi \sin\phi \, d\phi = -32\pi \cos\phi \Big]_{\phi=0}^{\phi=\pi}$$

$$= -32\pi \cos(\pi) + 32\pi \cos(0)$$

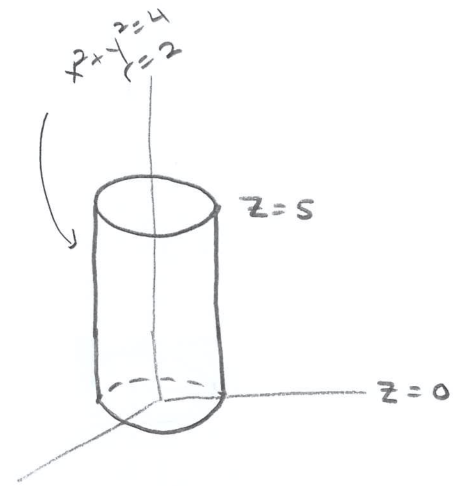
$$= -32\pi(-1) + 32\pi(1)$$

$$= \boxed{64\pi}$$

Problem 7

$$\vec{F} = \langle xy^2,\ yz^2,\ zx^2 \rangle$$

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = y^2 + z^2 + x^2$$



$$\iint_S \vec{F} \cdot ds = \iiint_E (x^2 + y^2 + z^2)\, dv$$

$0 \le z \le 5$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=0}^{z=5} (r^2 + z^2)\, r \, dz\, dr\, d\theta$$

$\rightarrow r^3 + rz^2$

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} r^3 z + \frac{1}{3} r z^3 \Big]_{z=0}^{z=5} dr\, d\theta$$

$5r^3 + \frac{125}{3} r$

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \left(5r^3 + \frac{125}{3} r\right) dr\, d\theta$$

$$\int_{\theta=0}^{\theta=2\pi} \frac{5r^4}{4} + \frac{125}{6} r^2 \Big]_{r=0}^{r=2} d\theta$$

$20 + \frac{250}{3} = \frac{310}{3}$

$$\int_{\theta=0}^{\theta=2\pi} \frac{310}{3}\, d\theta = \frac{310}{3}\theta \Big]_0^{2\pi} = \boxed{\frac{620\pi}{3}}$$