MATH 220 Exam 2 – Sample Test – Detailed Solutions

Problem 1 (B)

$$T(2\mathbf{u}-\mathbf{v}) = 2T(\mathbf{u}) - T(\mathbf{v})$$

$$2\begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}0\\5\end{bmatrix}$$

$$\begin{bmatrix}6\\2\end{bmatrix} - \begin{bmatrix}0\\5\end{bmatrix}$$

$$\begin{bmatrix}6\\-3\end{bmatrix}$$

Problem 2 (C)

A domain of $\mathbb{R}^3$ and codomain of $\mathbb{R}^2$ means that the transformation is going from $\mathbb{R}^3$ to $\mathbb{R}^2$.

We can immediately cross off answer choices A and B, since those are transforming elements from $\mathbb{R}^2$.

Next, we can cancel out answer choice D because the $x_1 - 5$ has a constant, so it is non-linear.

Therefore the answer must be C.

Problem 3 (D)
Put the matrix in echelon form and look for pivots:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{-R_1 + R_2, -R_1 + R_3, -R_1 + R_4}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\boldsymbol{-R_2 + R_3, -R_2 + R_4}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\boldsymbol{-R_3 + R_4}$$

$$\begin{bmatrix} \boxed{1} & 1 & 1 & 0 \\ 0 & 0 & \boxed{-1} & 1 \\ 0 & 0 & 0 & \boxed{-1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There is not a pivot in every row, and there is not a pivot in every column so it is neither.

Problem 4 (C)
Rotate counter clock-wise by $\pi/4$:

$$\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Reflect over x_1 axis:

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

which is equivalent to:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Problem 5 (D)

$$T\left(\begin{bmatrix}3\\2\end{bmatrix}\right) = \begin{bmatrix}4\\2\end{bmatrix}$$

$$\begin{bmatrix}1 & k\\0 & 1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$$

Multiplying the matrices gives us:

$$3 + 2k = 4$$
$$0 + 2 = 2$$

Solving the top equation gives us:

$$2k = 1$$

$$k = \frac{1}{2}$$

Problem 6 (A)

$$AB^T = \begin{bmatrix} -1 & 3 & -2 \\ 2 & 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1+6-10 & 2+3+0 \\ 2+8+0 & -4+4+0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 5 \\ 10 & 0 \end{bmatrix}$$

Problem 7

Set up an augmented matrix with the identity matrix. Then transform the original matrix using row operations until the original matrix is identity matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 4 & 6 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\boldsymbol{\frac{1}{4}R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\boldsymbol{-2R_2 + R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/2 & 0 \\ 0 & 1 & 3/2 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\boldsymbol{-\frac{3}{2}R_3 + R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/2 & 0 \\ 0 & 1 & 0 & 0 & 1/4 & -3/2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Problem 8

Start by finding B^{-1}:

$$B^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$B^{-1} = \frac{1}{(2)(2) - (3)(1)}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Now find AB:

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$$

Now add $B^{-1} + AB$:

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} + \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 4 \\ -10 & -3 \end{bmatrix}$$

Problem 9 (B)

Use row operations to get as many 0's as possible in a single row or column. I will work with the last column.

$$\begin{bmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{bmatrix}$$

$$\mathbf{-2R_1 + R_2}$$

$$\begin{bmatrix} 2 & 5 & 4 & 1 \\ 0 & -3 & -2 & 0 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{bmatrix}$$

Now do a cofactor expansion down the last column. Remember that sign on 1 will be negative because of its position.

$$-1\begin{vmatrix} 0 & -3 & -2 \\ 6 & -2 & -4 \\ -6 & 7 & 7 \end{vmatrix}$$

From here, you can continue with row operations, or just calculate the determinant as is. I will add $\mathbf{R_2 + R_3}$ to simplify things a little bit…

$$-1\begin{vmatrix} 0 & -3 & -2 \\ 6 & -2 & -4 \\ 0 & 5 & 3 \end{vmatrix}$$

Doing a cofactor expansion down the first column will give us:

$$(-1)[(0) - 6(-9 + 10) + 0]$$

$$-1(-6)$$

$$6$$

Problem 10 (C)

$$\det\left(\left(\frac{1}{2}A\right)^{-1} B^T C^3\right)$$

$$= \left(\det\left(\frac{1}{2}A\right)^{-1}\right)(\det B^T)(\det C^3)$$

$$= \left(\frac{1}{\det\left(\frac{1}{2}A\right)}\right)(\det B)(\det C)^3$$

$$= \left(\frac{1}{\left(\frac{1}{2}\right)^3 (\det A)}\right)(\det B)(\det C)^3$$

$$= \left(\frac{1}{\left(\frac{1}{8}\right)(4)}\right)(-3)(2)^3$$

$$= (2)(-3)(8) = -48$$

Problem 11 (A)

$$\det\begin{bmatrix} a & b & c \\ 3 & 4 & 5 \\ 2 & 3 & 9 \end{bmatrix} = 4$$

$$\det\begin{bmatrix} a & 3 & 2 \\ b & 4 & 3 \\ c & 5 & 9 \end{bmatrix} = 4$$

$$\det\begin{bmatrix} 3 & a & 2 \\ 4 & b & 3 \\ 5 & c & 9 \end{bmatrix} = -4$$

$$\det\begin{bmatrix} -3 & a & 2 \\ -4 & b & 3 \\ -5 & c & 9 \end{bmatrix} = -(-4) = 4$$

$$\det\begin{bmatrix} -3 & 5a & 2 \\ -4 & 5b & 3 \\ -5 & 5c & 9 \end{bmatrix} = (5)(4) = 20$$

$$\det\begin{bmatrix} -3 & 5a & 4 \\ -4 & 5b & 6 \\ -5 & 5c & 18 \end{bmatrix} = (3)(20) = 60$$

Problem 12

a) No. Does not contain the 0 vector
b) No. Not linear because of the $\sqrt{x_1}$
c) No. Not linear because of the x_1^2
d) Yes. Contains the 0 vector and is linear.

Problem 13
Getting A in echelon form gives us:

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Columns 1 and 2 have pivots, so columns 1 and 2 from the original matrix A form the basis for col$A$.

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

b)

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = -3x_2 - 3x_3 - 2x_4$$
$$x_2 = -2x_3 + x_4$$
$$x_3 = free$$
$$x_4 = free$$

$$x_1 = -3(-2x_3 + x_4) - 3x_3 - 2x_4 = 6x_3 - 3x_4 - 3x_3 - 2x_4$$

$$x_1 = 3x_3 - 5x_4$$
$$x_2 = -2x_3 + 1x_4$$
$$x_3 = 1x_3 + 0x_4$$
$$x_4 = 0x_3 + 1x_4$$

$$x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}; \text{ (} x_3 \text{ and } x_4 \text{ are free variables)}$$

Problem 14 (C)

a) **True**: $(A^T B^T)^{-1} = (B^T)^{-1}(A^T)^{-1} = (B^{-1})^T (A^{-1})^T$
b) **True**: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
c) **False**: $(AB)^2 = (AB)(AB) \neq B^2 A^2$
d) **True**: $(A^{-1}B)^{-1} = B^{-1}(A^{-1})^{-1} = B^{-1}A$