MATH141 Final Exam (Review of Ex 1, 2, & 3) – Sample Test – Detailed Solutions

Problem 1: B

Use integration by parts

$$u = \tan^{-1} x \qquad v = x$$

$$du = \frac{1}{1+x^2}\,dx \qquad dv = dx$$

Applying the formula gives us:

$$uv - \int v\,du$$

$$x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx$$

Use a u substitution to compute the remaining integral:

$$\int \frac{x}{1+x^2}\,dx$$

$$= \int \frac{x}{u}\left(\frac{du}{2x}\right)$$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln u$$

$$= \frac{1}{2}\ln(1+x^2)$$

$$u = 1 + x^2$$

$$du = 2x dx$$

$$dx = \frac{du}{2x}$$

Now evaluate from 0 to 1:

$$\left[x\tan^{-1} x - \frac{1}{2}\ln\,(1+x^2)\right]_0^1$$

$$\left[\tan^{-1}(1) - \frac{1}{2}\ln\,(2)\right] - \left[0 - \frac{1}{2}\ln(0)\right]$$

$$\frac{\pi}{4} - \frac{1}{2}\ln 2$$

Problem 2: D

Since all powers or sine and cosine are even, use the half-angle identities:

$$\int_0^{\pi/8} \frac{1}{2}(1+\cos 2x)\,\frac{1}{2}(1-\cos 2x)\,dx$$

$$\frac{1}{4}\int_0^{\pi/8} 1-\cos^2(2x)\,dx$$

Use the half angle identity again:

$$\frac{1}{4}\int_0^{\pi/8} \left(1-\frac{1}{2}(1-\cos(4x)\right)dx$$

$$\frac{1}{4}\int_0^{\pi/8} \left(\frac{1}{2}-\frac{1}{2}\cos(4x)\right)dx$$

$$\frac{1}{4}\left[\frac{1}{2}x-\frac{1}{8}\sin(4x)\right]_0^{\pi/8}$$

$$\frac{1}{4}\left[\frac{\pi}{16}-\frac{1}{8}\sin\left(\frac{\pi}{2}\right)\right]-\frac{1}{4}\left[0-\frac{1}{8}\sin(0)\right]$$

$$\frac{1}{4}\left[\frac{\pi}{16}-\frac{1}{8}\right]=\frac{1}{4}\left[\frac{\pi-2}{16}\right]=\frac{\pi-2}{64}$$

Problem 3: E

Use integration by parts, twice.

Apply integration by parts with:

$$u = x^2 \qquad v = \frac{1}{2}\sin(2x)$$

$$du = 2xdx \qquad dv = \cos(2x)\,dx$$

$$uv - \int v\,du$$

$$\frac{1}{2}x^2\sin\,(2x) - \int \frac{1}{2}2x\sin(2x)\,dx$$

$$\frac{1}{2}x^2\sin\,(2x) - \int x\sin(2x)\,dx$$

Now apply integration by parts with:

$$u = x \qquad v = -\frac{1}{2}\cos(2x)$$

$$du = dx \qquad dv = \sin(2x)\,dx$$

$$\frac{1}{2}x^2\sin\,(2x) - \left[uv - \int v\,du\right]$$

$$\frac{1}{2}x^2\sin(2x) - \left[-\frac{1}{2}x\cos(2x) + \int \frac{1}{2}\cos(2x)\,dx\right]$$

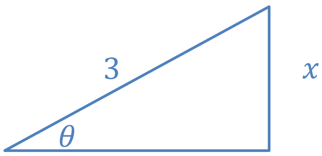
$$\frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \int \frac{1}{2}\cos(2x)\,dx$$

$$\frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin\,(2x) + C$$

Problem 4
This is a trig sub integral.

$$x = 3\sin\theta \rightarrow \sin\theta = \frac{x}{3} = \frac{opp}{hyp}$$

$$dx = 3\cos\theta\, d\theta$$



$$\int_0^3 \frac{x^2}{\sqrt{9-x^2}}\,dx$$

$$\int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}} 3\cos\theta\, d\theta = \int \frac{9\sin^2\theta}{\sqrt{9(1-\sin^2\theta)}} 3\cos\theta\, d\theta = \int \frac{9\sin^2\theta}{3\cos\theta} 3\cos\theta\, d\theta$$

$$\int 9\sin^2\theta\, d\theta$$

$$9\int \left(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right) d\theta$$

$$9\left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]$$

$$9\left[\frac{1}{2}\theta - \frac{1}{4}(2\sin\theta\cos\theta)\right]$$

$$9\left[\frac{1}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right)\right]_0^3$$

$$9\left[\frac{1}{2}\sin^{-1}(1) - \frac{1}{2}(1)(0)\right] - 9\left[\frac{1}{2}\sin^{-1}(0) - \frac{1}{2}(0)(1)\right]$$

$$9\left[\frac{1}{2}\left(\frac{\pi}{2}\right) - 0\right] - 0$$

$$\frac{9\pi}{4}$$

Problem 5
This is a partial fraction integral.

$$\frac{2x-3}{x(x^2+3)}=\frac{A}{x}+\frac{Bx+C}{x^2+3}$$

$$2x-3=A(x^2+3)+(Bx+C)x$$

Let $x=0$: $-3=A(3)\rightarrow$ $\boldsymbol{A=-1}$

$$2x-3=-1(x^2+3)+(Bx+C)x$$

$$2x-3=-x^2-3+Bx^2+Cx$$

$$2x-3=(-1+B)x^2+Cx-3$$

This gives us:

$$\begin{cases}-1+B=0\rightarrow \boldsymbol{B}=1\\ \boldsymbol{C=2}\end{cases}$$

$$\int\frac{2x-3}{x^3+3x}dx=\int\left(-\frac{1}{x}+\frac{x+2}{x^2+3}\right)dx$$

$$\int\left(-\frac{1}{x}+\frac{x}{x^2+3}+\frac{2}{x^2+3}\right)dx$$

$$\int-\frac{1}{x}dx+\int\frac{x}{x^2+3}dx+\int\frac{2}{x^2+3}dx$$

$$-\ln x+\frac{1}{2}\ln(x^2+3)+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+C$$

For $\int\frac{x}{x^2+3}dx$ use a $u-sub,$ where $u=x^2+3$

For $\int\frac{2}{x^2+3}dx$ pull out the 2 and recognize this as an arctan integral

Problem 6

Integrate both sides….

$$\int (2y-6)dy = \int (2-e^x)dx$$

$$y^2 - 6y = 2x - e^x + c$$

Plug in $y(0) = 0$ to solve for C:

$$0 = -1 + C$$
$$C = 1$$

$$y^2 - 6y = 2x - e^x + 1$$

Complete the square to solve for y:

$$y^2 - 6y + \boxed{9} = 2x - e^x + 1 + \boxed{9}$$

$$(y-3)^2 = 2x - e^x + 10$$

$$y - 3 = \pm\sqrt{2x - e^x + 10}$$

$$y = 3 \pm \sqrt{2x - e^x + 10}$$

Since we have to meet the initial value of $y(0) = 0$, then $y = 3 - \sqrt{2x - e^x + 10}$ is the only solution.

Problem 7

a) $P' = 4 - 0.10P$

b) $0 = 4 - 0.10P$
$.10P = 4$
$P = \frac{4}{.10} = 40$

c) Solve part A as a separable differential equation:

$$\frac{dP}{dt} = 4 - 0.10P$$

$$\frac{1}{4 - 0.10P}dP = dt$$

$$\frac{\ln(4 - 0.10P)}{-0.10} = t + C$$

$$\ln(4 - 0.10P) = -0.10t + C$$

$$4 - 0.10P = Ce^{-0.10t}$$

$$0.10P = 4 - Ce^{-0.10t}$$

$$P = \frac{4 - Ce^{-0.10t}}{0.10}$$

$$P = 40 - Ce^{-0.10t}$$

Use $P(0) = 50$ to solve for C:

$50 = 40 - Ce^0$
$10 = -C$
$C = -10$

$$P = 40 + 10e^{-0.10t}$$

d) 40,000 fish

Problem 8: C
Use L'Hospital's Rule:

$$\lim_{x \to 1} \frac{e^x}{\frac{1}{x}} = \frac{e}{1} = e$$

Problem 9: D
Evaluate the limit using L'Hospital's Rule.

Simplify first…

$$e^{\lim_{x \to 0} \frac{\ln(\sin x + e^x)}{x}}$$

Apply L.H. Rule…

$$e^{\lim_{x \to 0} \frac{\frac{1}{\sin x + e^x}(\cos x + e^x)}{1}}$$

$$e^{\lim_{x \to 0} \frac{\cos x + e^x}{\sin x + e^x} = e^2}$$

Problem 10: C

Given $\sum_{n=1}^{\infty} a_n = \pi$, we can conclude that:

$$\lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \lim_{n\to\infty} s_n = \pi$$

Therefore:

$$\lim_{n\to\infty} \left(e^{a_n} - 4\cos(s_n)\right) = e^0 - 4\cos(\pi) = 1 - 4(-1) = 5$$

Problem 11: B

$$\sum_{n=0}^{\infty} \frac{4}{5^{n+1}} + \sum_{n=0}^{\infty} \frac{(-2)^n}{5^{n+1}}$$

$$\sum_{n=0}^{\infty} \frac{4}{5 \cdot 5^n} + \sum_{n=0}^{\infty} \frac{(-2)^n}{5 \cdot 5^n}$$

$$\sum_{n=0}^{\infty} \left(\frac{4}{5}\right)\left(\frac{1}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)\left(\frac{-2}{5}\right)^n$$

$c = \frac{4}{5}$, $r = \frac{1}{5}$ + $c = \frac{1}{5}$, $r = -\frac{2}{5}$

converges to...

$$\frac{\frac{4}{5}}{1 - \frac{1}{5}}$$

$$= 1$$

converges to

$$\frac{\frac{1}{5}}{1 + \frac{2}{5}} = \frac{\frac{1}{5}}{\frac{7}{5}}$$

$$= \frac{1}{7}$$

$$1 + \frac{1}{7} = \boxed{\frac{8}{7}}$$

Problem 12: E

$$\sum_{n=0}^{\infty} (-1)^n \frac{(27)^n}{(16)^n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{-27}{16}\right)^n$$

$$|r| = \frac{27}{16} > 1$$

$$\boxed{\text{Diverges by Geometric Series}}$$

Problem 13
This is an improper integral with integration by parts…

$$\int_{-\infty}^{0} xe^x\,dx$$

$$\lim_{t\to-\infty} \int_{t}^{0} xe^x\,dx$$

Use integration by parts…

$$u = x \qquad v = e^x$$

$$du = dx \qquad dv = e^x dx$$

$$\lim_{t\to-\infty} \left[uv - \int v\,du\right]$$

$$\lim_{t\to-\infty} \left[xe^x - \int e^x\,dx\right]$$

$$\lim_{t\to-\infty} [xe^x - e^x]_t^0$$

$$\lim_{t\to-\infty} [(0 - e^0) - (te^t - e^t)]$$

$$\lim_{t\to-\infty} [-1 - te^t + e^t] = -1 - 0 + 0 = -1$$

Problem 14

I. Diverges

$$\lim_{n\to\infty}\left|(-1)^n\left(\frac{2n^2+1}{3n^2+n}\right)\right| = \frac{2}{3} \neq 0$$

Since this is an alternating sequence, whose lim $\neq 0$, the sequence $\boxed{\text{diverges}}$

II. Converges to e^4

$$\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n = 1^\infty \to \text{indeterminate}$$

$$e^{\lim_{n\to\infty}\ln\left(1+\frac{4}{n}\right)^n}$$

$$= e^{\lim_{n\to\infty}\frac{\ln\left(1+\frac{4}{n}\right)}{\frac{1}{n}}}$$

L.R. $\rightarrow$

$$e^{\lim_{n\to\infty}\frac{\frac{1}{1+\frac{4}{n}}\left(-\frac{4}{n^2}\right)}{-\frac{1}{n^2}}} = e^{\lim_{n\to\infty}\frac{4}{1+\cancel{\frac{4}{n}}^{\,0}}} = e^4$$

$$\boxed{\text{converges to } e^4}$$

III. Converges to $\pi/2$

$$\underbrace{\lim_{n\to\infty} \tan^{-1}(n)}_{\pi/2} + \underbrace{\lim_{n\to\infty} \left| (-1)^n \frac{\ln(n)}{n^2} \right|}_{0}$$

$$\boxed{\text{converges to } \frac{\pi}{2}}$$

IV. Converges to 0

$$\lim_{n\to\infty} \frac{2^n}{n!} = \lim_{n\to\infty} \frac{2\cdot 2\cdot 2\cdot \cdots \cdot 2}{1\cdot 2\cdot 3 \cdots (n-2)(n-1)(n)} = 0 \quad \text{(by rates of growth)}$$

$$\boxed{\text{converges to } 0}$$

Problem 15

a) We need $\int_{-\infty}^{\infty} p(x)\,dx = 1.$

$$\cancel{\lim_{t\to-\infty}\int_t^0 0\,dx} \to 0 + \lim_{t\to\infty}\int_0^t ce^{-\frac{1}{20}x}\,dx = 1$$

$$\lim_{t\to\infty} -20ce^{\frac{-1}{20}x}\Big|_0^t = 1$$

$$\lim_{t\to\infty}\left(\cancel{-20ce^{\frac{-1}{20}t}} \to 0 + 20ce^{\frac{-1}{20}(0)}\right) = 1$$

$$20ce^0 = 1$$

$$\boxed{c = \frac{1}{20}}$$

b) $\int_5^{10} \frac{1}{20}e^{\frac{-1}{20}x}\,dx$

$$= -20\left(\frac{1}{20}e^{-1/20x}\right)\Big|_5^{10}$$

$$= -e^{-10/20} + e^{-5/20}$$

$$= \boxed{e^{-1/4} - e^{-1/2}}$$

c) $\lim\limits_{t \to \infty} \int_{30}^{t} \frac{1}{20} e^{-\frac{1}{20}x} \, dx$

$\lim\limits_{t \to \infty} -e^{-\frac{1}{20}x} \Big|_{30}^{t}$

$\lim\limits_{t \to \infty} \left(\cancelto{0}{-e^{-\frac{1}{20}t}} + e^{-\frac{30}{20}} \right) = \boxed{e^{-1.5}}$

d) $\text{mean} = \int_{-\infty}^{\infty} x \cdot p(x) \, dx$

$\cancelto{0}{\lim\limits_{t \to -\infty} \int_{t}^{0} 0 \, dx} + \lim\limits_{t \to \infty} \int_{0}^{t} x \cdot \frac{1}{20} e^{-x/20} \, dx$

$\lim\limits_{t \to \infty} \int_{0}^{t} \frac{1}{20} x e^{-x/20} \, dx \quad \rightarrow$

I.B.P.

$u = \frac{1}{20} x \qquad v = -20e^{-x/20}$

$du = \frac{1}{20} dx \qquad dv = e^{-x/20} \, dx$

$uv - \int v \, du$

$-xe^{-x/20} + \int e^{-x/20} \, dx$

$\lim\limits_{t \to \infty} \left[-xe^{-x/20} - 20e^{-x/20} \right] \Big|_{0}^{t}$

$$\lim_{t\to\infty}\left[\left(-\cancel{te^{-t/20}}^{\,0} - 20\cancel{e^{-t/20}}^{\,0}\right) - \left(0 - 20e^{0}\right)\right] = \boxed{20 \text{ min}}$$

$*$

$$* \lim_{t\to\infty} -te^{-t/20}$$

$$\lim_{t\to\infty} \frac{-t}{e^{t/20}} = 0 \quad \text{by rates of growth}$$

Problem 16

a) $A_1 = 30$

$A_2 = 30 + 30(.10)$

$A_3 = 30 + 30(.10) + 30(.10)^2$

b) $A_n = 30 + 30(.10) + 30(.10)^2 + \cdots + 30(.10)^{n-1}$

$$\boxed{A_n = \frac{30 - 30(.10)^n}{1 - .10}}$$

c) $$\lim_{n\to\infty} \frac{30 - \cancel{3(.10)^n}^{\,0}}{.9} = \frac{30}{.9} = 30 \cdot \frac{10}{9} = \frac{300}{9} = \frac{100}{3} \approx 33.3$$

$\boxed{\text{No}}$

Problem 17

$$\lim_{n\to\infty} (-1)^n \frac{2n^3+5}{3n^3+2n-1} \neq 0$$

$\boxed{\text{Diverges by Test For Divergence}}$

Problem 18

compare to $\sum \frac{2}{n^5}$ → conv. by p-series

$$\frac{2}{n^5} \geq \frac{1+\sin^2(n)}{n^5}$$ → True ✓

$\boxed{\text{converges by Direct Comparison Test}}$

Problem 19

First look at $\sum |a_n|$:

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^{3/4}} \rightarrow \text{Integral Test}$$

$f(x) = \dfrac{1}{x(\ln(x))^{3/4}}$ is cont., positive, & decreasing on $(2, \infty)$

$$\lim_{t\to\infty} \int_2^t \frac{1}{x(\ln(x))^{3/4}}\,dx$$

$\rightarrow$ u-sub

$u = \ln(x)$

$$\lim_{t\to\infty} \left. 4(\ln(x))^{1/4} \right]_2^t$$

$$\lim_{t\to\infty} \left[4(\ln t)^{1/4} - 4(\ln 2)^{1/4}\right] = \infty - \# = \infty$$

So $\sum |a_n|$ diverges.

Now look at $\sum a_n$:

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n(\ln(n))^{3/4}} \rightarrow \text{Alternating Series Test}$$

$$b_n = \frac{1}{n(\ln(n))^{3/4}}$$

1. $\displaystyle \lim_{n\to\infty} \frac{1}{n(\ln(n))^{3/4}} = 0$

2. $b_{n+1} \leq b_n$

So $\sum a_n$ converges.

$\boxed{\text{conditionally convergent}}$

Problem 20

Root Test

$$\lim_{n \to \infty} \sqrt[n]{\left(\frac{2n^3 + 5n - 2}{5n^3 + 10}\right)^{2n}}$$

$$\lim_{n \to \infty} \left(\frac{2n^3 + 5n - 2}{5n^3 + 10}\right)^2 = \left(\frac{2}{5}\right)^2 = \frac{4}{25} < 1$$

$\boxed{\text{absolutely convergent}}$

Problem 21

Test For Divergence

$$\lim_{n \to \infty} \frac{2n+7}{\sqrt{n^2+1}} = 2, \quad \text{so } \sum |a_n| \text{ diverges}$$

$$\lim_{n \to \infty} (-1)^{n-1} \left(\frac{2n+7}{\sqrt{n^2+1}}\right) = \text{DNE}, \quad \text{so } \sum a_n \text{ diverges}$$

$\boxed{\text{divergent}}$

Problem 22

Ratio Test

$$\lim_{n\to\infty} \left| \frac{e^{n+1}((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{e^n\, n!} \right|$$

$$\lim_{n\to\infty} \left| \frac{\cancel{e^n} \cdot e\, ((n+1)\cdot \cancel{n!})^2\ \cancel{(2n)!}}{(2n+2)(2n+2)\,\cancel{(2n)!}\ \cancel{e^n}\,\cancel{(n!)^2}} \right|$$

$$\lim_{n\to\infty} \left| \frac{e\,(n+1)^2}{(2n+2)(2n+2)} \right| = \frac{e}{4} < 1$$

$\boxed{\text{absolutely convergent}}$

Problem 23:

Use the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(3x-2)^{n+1}}{(n+1)3^{n+1}}\times\frac{n\,3^n}{(3x-2)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(3x-2)n}{3(n+1)}\right|=\left|\frac{3x-2}{3}\right|$$

$$-1<\left|\frac{3x-2}{3}\right|<1$$

$$-3<3x-2<3$$

$$-1<3x<5$$

$$-\frac{1}{3}<x<\frac{5}{3}$$

Radius: $\mathbf{\frac{1}{2}\left(\frac{5}{3}+\frac{1}{3}\right)=1}$

Test Endpoints:

$$x=-\frac{1}{3}:\ \sum_{n=1}^{\infty}(-1)^n\frac{(-3)^n}{n\,3^n}=\sum_{n=1}^{\infty}\frac{3^n}{n\,3^n}=\sum_{n=1}^{\infty}\frac{1}{n}\ \text{diverges by harmonic series}$$

$$x=\frac{5}{3}:\ \sum_{n=1}^{\infty}(-1)^n\frac{(3)^n}{n\,3^n}=\sum_{n=1}^{\infty}\frac{(-3)^n}{n\,3^n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\ \text{conveges by alternating series test}$$

Interval: $\left(-\frac{1}{3},\frac{5}{3}\right]$

Problem 24

Use the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{2^{n+1}(x+2)^{n+1}}{(n+2)!}\times\frac{(n+1)!}{2^n(x+2)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{2(x+2)}{(n+2)}\right|=0$$

Radius: ∞**, Interval:** $(-\infty,\infty)$

Problem 25

Use the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(4x-5)^{2n+2}}{\sqrt{n+1}\,(9)^{n+1}}\times\frac{\sqrt{n}\,(9)^n}{(4x-5)^{2n}}\right|$$

$$\lim_{n\to\infty}\left|\frac{(4x-5)^2\sqrt{n}}{9\sqrt{n+1}}\right| = \left|\frac{(4x-5)^2}{9}\right|$$

$$\left|\frac{(4x-5)^2}{9}\right| < 1$$

$$|(4x-5)^2| < 9$$

$$|4x-5| < 3$$

$$\left|x-\frac{5}{4}\right| < \frac{3}{4}$$

Radius: $3/4$