MATH141 Final Exam (New Material) – Sample Test Solutions

Problem 1: D

Algebraically manipulate the function to get:

$$\frac{x}{2\left(1-\frac{x^3}{2}\right)} = \frac{x}{2}\frac{1}{\left(1-\frac{x^3}{2}\right)}$$

$$= \frac{x}{2}\sum_{n=0}^{\infty}\left(-\frac{x^3}{2}\right)^n$$

$$= \frac{x}{2}\sum_{n=0}^{\infty}(-1)^n\frac{x^{3n}}{2^n}$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+1}}{2^{n+1}}$$

Problem 2: C

Get the function in terms of a power series, then integrate:

$$\int 2\frac{1}{1-(-4x^2)}\,dx = \int 2\sum_{n=0}^{\infty}(-4x^2)^n\,dx$$

$$\int 2\sum_{n=0}^{\infty}(-1)^n(4)^n(x)^{2n}\,dx = \int 2\sum_{n=0}^{\infty}(-1)^n(2)^{2n}(x)^{2n}\,dx = \int \sum_{n=0}^{\infty}(-1)^n(2)^{2n+1}(x)^{2n}\,dx$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{(2)^{2n+1}(x)^{2n+1}}{2n+1} + C$$

Problem 3: B

Use the Maclaurin series for $f(x) = \ln(1 + x)$ as your base series.

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$$

$$f(x) = x^2 \ln\left(1 + \frac{x}{3}\right)$$

$$f(x) = x^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\frac{x}{3}\right)^n}{n}$$

$$f(x) = x^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{(3)^n n}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{n+2}}{(3)^n n}$$

Problem 4: D

Use the Maclaurin series for $f(x) = \cos x$ backwards:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{3^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{\left(-\frac{\pi}{3}\right)^{2n}}{(2n)!} = \cos\frac{\pi}{3} = \frac{1}{2}$$

Problem 5: A

Use the Maclaurin series for $f(x) = e^x$, and then extract the first two terms to solve for the series that starts at $n = 2$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty} \frac{1^n}{2^n n!} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n!} = e^{1/2}$$

$$e^{1/2} = \frac{\left(\frac{1}{2}\right)^0}{0!} + \frac{\left(\frac{1}{2}\right)^1}{1!} + \sum_{n=2}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n!}$$

$$e^{1/2} = 1 + \frac{1}{2} + \sum_{n=2}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n!}$$

$$\sum_{n=2}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n!} = \boldsymbol{e^{1/2} - \frac{3}{2}}$$

Problem 6: A

$$\int xe^{-2x}\,dx = \int x\sum_{n=0}^{\infty}\frac{(-2x)^n}{n!}\,dx$$

$$\int x\sum_{n=0}^{\infty}(-1)^n\frac{2^n x^n}{n!}\,dx = \int\sum_{n=0}^{\infty}(-1)^n\frac{2^n x^{n+1}}{n!}\,dx$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{2^n x^{n+2}}{(n+2)(n!)} + C$$

Problem 7: C

$$f'(x) = -2\sin(2x)$$
$$f''(x) = -4\cos(2x)$$
$$f'''(x) = 8\sin(2x)$$
$$f^{(4)}(x) = 16\cos(2x)$$

$$M = \text{Max value of } |16\cos(2x)| \text{ on } [0, \pi]$$
$$M = 16$$

$$R(x) \le \frac{M}{(n+1)!}\left(\left|x - \frac{\pi}{2}\right|^{n+1}\right)$$

$$R(x) \le \frac{16}{24}\left(\frac{\pi}{2}\right)^4 = \frac{\pi^4}{24}$$

Problem 8: B

$$2e^{\frac{3\pi}{4}i} = 2\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$

$$2e^{\frac{3\pi}{4}i} = 2\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$2e^{\frac{3\pi}{4}i} == -\sqrt{2} + \sqrt{2}i$$

Problem 9: D

$$3ie^{\frac{4\pi}{3}i} = 3i\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$$

$$3ie^{\frac{4\pi}{3}i} = 3i\left(-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$3ie^{\frac{4\pi}{3}i} = -\frac{3}{2}i + i^2\left(-\frac{3\sqrt{3}}{2}\right)$$

$$3ie^{\frac{4\pi}{3}i} = -\frac{3}{2}i + (-1)\left(-\frac{3\sqrt{3}}{2}\right)$$

$$3ie^{\frac{4\pi}{3}i} = -\frac{3}{2}i + \frac{3\sqrt{3}}{2}$$

Problem 10

a) $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$
$y' = a_1 + 2a_2 x + 3a_3 x^2$

$$\frac{dy}{dx} = -y$$

$$a_1 + 2a_2 x + 3a_3 x^2 = -a_0 - a_1 x - a_2 x^2 - a_3 x^3$$

This gives us the equations below by equating the coefficients:

$$a_1 = -a_0$$

$$2a_2 = -a_1$$

$$3a_3 = -a_2$$

So

$a_0 = 1$ (from the problem)

$$a_1 = -1$$

$$a_2 = -\frac{1}{2}a_1 = \frac{1}{2}$$

$$a_3 = -\frac{1}{3}a_2 = -\frac{1}{6}$$

b) Now use the power series definitions:

$$y = \sum_{n=0}^{\infty} a_n x^n \,, y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = -\sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [x^n((n+1)a_{n+1} + a_n)] \;\; 0$$

$$(n+1)a_{n+1} + a_n = 0$$

$$a_{n+1} = -\frac{a_n}{n+1}$$

$$a_0 = 1$$

$$a_1 = -\frac{a_0}{0+1} = -\frac{1}{1} = -1$$

$$a_2 = -\frac{a_1}{1+1} = -\frac{-1}{2} = \frac{1}{2}$$

$$a_3 = -\frac{a_2}{2+1} = -\frac{1}{2(3)} = -\frac{1}{6}$$

$$a_n = \frac{(-1)^n}{n!}$$

c) Now solve use separation of variables:

$$\frac{dy}{dx} = -y$$

$$\frac{1}{y}dy = -1dx$$

$$\ln|y| = -x + C$$

$$y = Ce^{-x}$$

Solve for C using $y(0) = 1$:

$$1 = C$$

$$y = e^{-x}$$

Maclaurin Series for $y = e^{-x}$ is:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

Problem 11

Start by multiplying both sides by r:

$$(r)r = 6(r)\cos\theta$$

$$r^2 = 6r\cos\theta$$

$$x^2 + y^2 = 6x$$

$$x^2 - 6x + \boxed{9} + y^2 = \boxed{9}$$

$$(x-3)^2 + y^2 = 9$$

Problem 12

Use the formulas to convert from Cartesian to Polar:

$$y^2 = 4x$$

$$(r\sin\theta)^2 = 4r\cos\theta$$

$$r^2\sin^2\theta = 4r\cos\theta$$

$$r\sin^2\theta = 4\cos\theta$$

$$r = \frac{4\cos\theta}{\sin^2\theta} = 4\frac{\cos\theta}{\sin\theta}\frac{1}{\sin\theta} = 4\cot\theta\csc\theta$$

Problem 13

$$1 \le r < 2$$

$$\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$$