Problem 1

$$\boldsymbol{Answer}: \boldsymbol{V(t)} = \begin{cases} \mathbf{70,000 - 6000}\boldsymbol{t}, & \boldsymbol{t} \leq \mathbf{5} \\ \mathbf{40,000(.93)}^{\boldsymbol{t}-\mathbf{5}}, & \boldsymbol{t} > \mathbf{5} \end{cases}$$

First 5 years:
Linear Function
$b = 70{,}0000$
$m = -6{,}000$

After 5 years:
Exponential Decay
$A = 70{,}000 - 6{,}000(5) = 40{,}000$
$b = 1 - .07 = 0.93$

Problem 2

$\boldsymbol{Answer}$: $\boldsymbol{D(t) = 4 + 2\cos\left(\frac{\pi}{6}(t-7)\right)}$

$$M = \frac{2+6}{2} = 4$$

$$A = \frac{6-2}{2} = 2$$

$$P = 12$$

$$C = 7$$

Use the cosine function since you are given a data point about when high-tide (the maximum) occurs.

$$D(t) = M + A\cos\left(\frac{2\pi}{P}(t-c)\right)$$

$$D(t) = 4 + 2\cos\left(\frac{2\pi}{12}(t-7)\right)$$

Problem 3

Answer: $\mathbf{17}\ \boldsymbol{ft/s}$

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{s(5) - s(2)}{5 - 2}$$

$$\frac{(3(5)^2 - 5) - (3(2)^2 - 2)}{5 - 2} = \frac{70 - 10}{5 - 2} = \frac{60}{3} = 20\ ft/s$$

Problem 4

Answer: $-\infty$

When you plug in you get 0/0, so simplify by factoring

$$\lim_{x \to 5^+} \frac{25 - x^2}{(5 - x)^2} = \lim_{x \to 5^+} \frac{(5 - x)(5 + x)}{(5 - x)(5 - x)} = \lim_{x \to 5^+} \frac{(5 + x)}{(5 - x)}$$

Now when you plug in you get #/0, so test a value on the right side of 5 to see if the limit approaches positive infinity or negative infinity…

$$\text{Try } 5.1 \rightarrow \ \frac{5.1 + 5}{5 - 5.1} = \frac{+}{-} = -\infty$$

Problem 5

Answer: $\mathbf{7/4}$

When you plug in you get 0/0, so simplify by factoring

$$\lim_{x \to 3} \frac{2x^2 - 5x - 3}{x^2 - x - 6} = \lim_{x \to 3} \frac{(2x + 1)(x - 3)}{(x - 3)(x + 2)} = \lim_{x \to 3} \frac{(2x + 1)}{(x + 2)}$$

Now when you plug in you get…

$$\frac{2(3) + 1}{3 + 2} = \frac{7}{5}$$

Problem 6

Answer: -14

Since this is an absolute value limit, start by determining if what's inside the absolute value will be positive or negative as $x \to 7$ from the right.

$|7 - x|$ will be negative as $x \to 7$ from the right, so to remove the absolute value signs, we must insert a negative…

$$\lim_{x \to 7^+} \frac{14x - 2x^2}{-(7 - x)} = \lim_{x \to 7^+} \frac{2x(7 - x)}{-(7 - x)} = \lim_{x \to 7^+} -2x = -14$$

Problem 7

Answer: 1

When you evaluate the limit you get:

$$\frac{\lim_{x \to 3}(f(x) - 1)}{0} = 5$$

Since this limit evaluates to a number, that means it must be a 0/0 limit. So the top of the fraction above must equal 0.

$$\lim_{x \to 3}(f(x) - 1) = 0$$

$$\lim_{x \to 3} f(x) = 1$$

Problem 8

Answer: 0

When you evaluate the limit you get:

$$\frac{5}{\lim_{x \to 2} g(x)} = \infty$$

Since this limit evaluates to ∞, that means it must be a #/0 limit. So the bottom of the fraction must equal 0.

$$\lim_{x \to 2} g(x) = 0$$

Problem 9

$f(20) = f(15) + 10$

The population of Happy Valley in 2020 is 10,000 people more than the population of Happy Valley in 2015.

$f^{-1}(2) = 5$

The population of Happy Valley in 2005 is 17,000 people.

$f'(21) = 3$

In 2021, the population of Happy Valley is increasing by approximately 3000 people/year.

Problem 10

Answer:

a) $-7, 1$

b) $-7, -4, 1$

c) $-7, -4, -2, 1, 5$

Problem 11

Point: (0,2)
Slope: 2

To find the slope, take the derivative and plug in $x = 0$.

$$f'(x) = e^x(-6x^2) + (2 - 2x^3)e^x$$

$$f'(0) = 0 + 2 = 2$$

Equation of the tangent line:

$$y - 2 = 2(x - 0)$$

$$\boldsymbol{y = 2x + 2}$$

Problem 12

Find where acceleration is 7, then plug that into the velocity equation.

$$v(t) = t^2 - t + 1$$

$$a(t) = 2t - 1$$

$$7 = 2t - 1$$

$$t = 4$$

$$\boldsymbol{v(4) = (4)^2 - 4 + 1 = 13\ \mathrm{m/s}}$$

Problem 13

Simplify first using rules of logs, then take the derivative using implicit differentiation.

$$\ln(x) + \ln(y) + y^3 = 8$$

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

Now plug in $\left(\frac{1}{2}, 2\right)$.

$$\frac{1}{1/2} + \frac{1}{2}\frac{dy}{dx} + 3(2)^2\frac{dy}{dx} = 0$$

$$2 + \frac{1}{2}\frac{dy}{dx} + 12\frac{dy}{dx} = 0$$

$$2 + \frac{25}{2}\frac{dy}{dx} = 0$$

$$\frac{25}{2}\frac{dy}{dx} = -2$$

$$\frac{dy}{dx} = -\frac{4}{25}$$

Problem 14
Use logarithmic differentiation.

$$\ln(y) = \ln(\sin(x))^{x^2}$$

$$\ln(y) = x^2 \ln(sin(x))$$

$$\frac{1}{y}\frac{dy}{dx} = x^2 \frac{1}{\sin(x)}\cos(x) + \ln(sin(x))\,(2x)$$

$$\frac{dy}{dx} = (x^2\cot(x) + 2x\ln(sin\,x))y$$

$$\frac{dy}{dx} = (x^2\cot(x) + 2x\ln(sin\,x))\,(\sin x)^{x^2}$$

Problem 15

a. Use product rule overall.

$$g'(x) = f(x)e^{x^2}(2x) + e^{x^2}f'(x)$$

$$g'(1) = f(1)e^1(2) + e^1 f'(1)$$

$$g'(1) = 10(e)(2) + e(-4) = 20e - 4e = 16e$$

b. Simplify first using rules of logs, then take the derivative using log rules

$$i(x) = \ln(x) + \ln\big(f(x)\big)$$

$$i'(x) = \frac{1}{x} + \frac{1}{f(x)}f'(x)$$

$$i'(2) = \frac{1}{2} + \frac{f'(2)}{f(2)}$$

$$i'(2) = \frac{1}{2} + \frac{-5}{12}$$

$$i'(2) = \frac{1}{12}$$

c. Use the chain rule overall.

$$k'(x) = f'(e^{2x}) \cdot (e^{2x}) \cdot (2)$$

$$k'(0) = f'(e^0) \cdot (e^0) \cdot (2)$$

$$k'(0) = f'(1) \cdot (1) \cdot (2)$$

$$k'(0) = -8$$

Problem 16
Related Rates

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi(10)(10)$$

$$\frac{dA}{dt} = \boxed{200\pi \ \text{cm}^3/\text{s}}$$

$A = 100\pi$

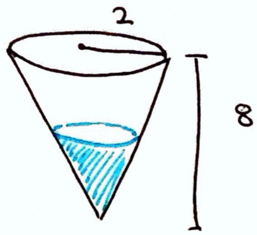
$r = 10$

$\frac{dA}{dt} = ?$

$\frac{dr}{dt} = 10$

$A = \pi r^2$
$100\pi = \pi r^2$
$r^2 = 100$
$r = 10$

Problem 17
Related Rates



$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi \left(\frac{1}{4}h\right)^2 h$

$V = \frac{1}{3} \cdot \frac{1}{16}\pi h^3$

$\frac{dv}{dt} = \frac{1}{16}\pi h^2 \frac{dh}{dt}$

$12 = \frac{1}{16}\pi (4)^2 \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{12}{\pi}$ ft/s

$l =$

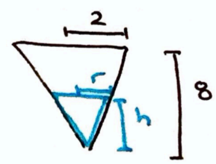
$r =$

$h = 4$

$\frac{dv}{dt} = 12$

$\frac{dr}{dt} =$ → missing, so eliminate r from equation

$\frac{dh}{dt} = ?$



$\frac{2}{8} = \frac{r}{h}$

$2h = 8r$

$r = \frac{1}{4}h$

Problem 18

$f(x) = \sqrt{x}, x = 16$

Point: (16,4)
Slope: 1/8

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8}$$

$$y - 4 = \frac{1}{8}(x - 16)$$

$$y = 4 + \frac{1}{8}(x - 16)$$

$$L(x) = 4 + \frac{1}{8}(x - 16)$$

$$L(15.8) = 4 + \frac{1}{8}(15.8 - 16)$$

$$L(15.8) = 4 + \frac{1}{8}(-0.2)$$

$$L(15.8) = 4 - \frac{0.2}{8} = 4 - 0.025 = 3.975$$

Problem 19

Note that f is continuous and differentiable so the MVT can be applied.

$$\frac{f(2) - f(-1)}{2 - (-1)} = f'(x)$$

$$\frac{(7 - (2)^2) - (7 - (1)^2)}{2 + 1} = f'(x)$$

$$\frac{3 - 6}{2 + 1} = f'(x)$$

$$f'(x) = -1$$

$$-2x = -1$$

$$x = 1/2$$

Problem 20

Find the critical numbers within the interval…

$$f'(x) = 1 - \frac{4}{x^2}$$

$$f'(x) = \frac{x^2}{x^2} - \frac{4}{x^2}$$

$$f'(x) = \frac{x^2 - 4}{x^2}$$

$$x^2 - 4 = 0, \qquad x^2 = 0$$

$$x = 2, \qquad x = -2, \qquad x = 0$$

$x = 2$ is the only critical number within the interval

Take $x = 2$, and the endpoints of the interval, $x = 1, x = 8$ and plug them into $f(x)$.

$$f(1) = 1 + \frac{4}{1} = 5$$

$$f(2) = 2 + \frac{4}{2} = 4$$

$$f(8) = 8 + \frac{4}{8} = 8.5$$

Absolute maximum value: 8.5
Absolute minimum value: 4

Problem 21

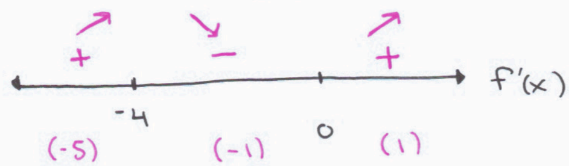
a. Take the derivative and set up a first derivative number line…

$$f'(x) = \frac{5}{3}x^{2/3} + \frac{20}{3}x^{-1/3}$$

$$f'(x) = \frac{5}{3}x^{2/3}\left(\frac{x^{1/3}}{x^{1/3}}\right) + \frac{20}{3x^{1/3}}$$

$$f'(x) = \frac{5x + 20}{3x^{1/3}} \rightarrow 5x + 20 = 0, \quad x = -4$$

$$\hookrightarrow 3x^{1/3} = 0, \quad x = 0$$



Critical Numbers:
$x = -4$, local max
$x = 0$, local min

b. Take the second derivative and set up a second derivative number line…

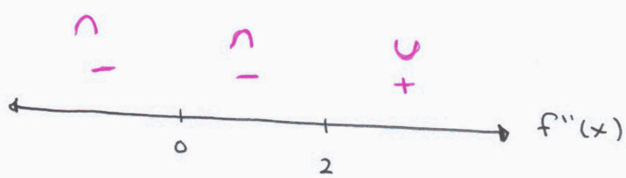
$$f'(x) = \frac{5}{3}x^{2/3} + \frac{20}{3}x^{-1/3}$$

$$f''(x) = \frac{10}{9}x^{-1/3} - \frac{20}{9}x^{-4/3}$$

$$f''(x) = \frac{10}{9x^{1/3}}\left(\frac{x}{x}\right) - \frac{20}{9x^{4/3}}$$

$$f''(x) = \frac{10x - 20}{9x^{4/3}} \rightarrow 10x - 20 = 0, \quad x = 2$$

$$\hookrightarrow 9x^{4/3} = 0, \quad x = 0$$



Concave up: $(2, \infty)$

c. Inflection Point: $x = 2$ only

Problem 22

In this problem we are trying to maximize distance:

$$d = \sqrt{(x-2)^2 + (y-0)^2}$$

$$D = (x-2)^2 + y^2$$

Constraint:

$$8x^2 + y^2 = 56$$

$$y^2 = 56 - 8x^2$$

$$D = (x-2)^2 + 56 - 8x^2$$

$$D' = 2(x-2) - 16x$$

$$D' = 2x - 4 - 16x$$

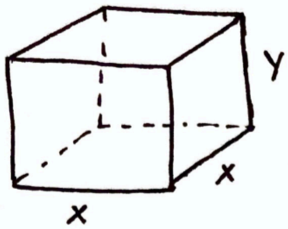
$$0 = -14x - 4$$

$$14x = -4$$

$$x = -\frac{4}{14} \rightarrow \boxed{x = -\frac{2}{7}}$$

Problem 23

In this problem we are trying to minimize cost:



$V = 8$

$x \cdot x \cdot y = 8$

$y = \frac{8}{x^2}$

$C = 5x^2 + 4(3)xy + 1x^2$

$C = 6x^2 + 12xy$

$C = 6x^2 + 12x\left(\frac{8}{x^2}\right)$

$C = 6x^2 + 96x^{-1}$

$C' = 12x - 96x^{-2}$

$0 = 12x - \frac{96}{x^2}$

$12x^3 = 96$

$x^3 = 8$

$x = 2 \rightarrow y = \frac{8}{x^2} = \frac{8}{4} = 2$

Dimensions: 2 m × 2m × 2m

Verify: Second Derivative Test

$C' = 12x - 96x^{-2}$

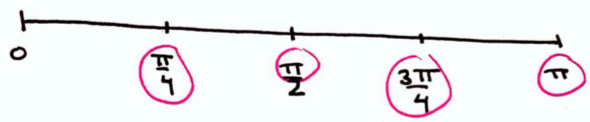
$C'' = 12 + \frac{96}{x^3}$

$C''(2) = 12 + \frac{96}{8} > 0$

Since $C'(2) = 0$, $C''(2) > 0$ then $x = 2$ is a min.

Problem 24

$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$$



$$A \approx \frac{\pi}{4}\left(f\left(\frac{\pi}{4}\right) + f(\pi) + f\left(\frac{3\pi}{4}\right) + f(\pi)\right)$$

$$\approx \frac{\pi}{4}\left(\sin^2\left(\frac{\pi}{4}\right) + \sin^2\left(\frac{\pi}{2}\right) + \sin^2\left(\frac{3\pi}{4}\right) + \sin^2(\pi)\right)$$

$$\approx \frac{\pi}{4}\left(\frac{1}{2} + 1 + \frac{1}{2} + 0\right)$$

$$\approx \frac{\pi}{4}(2)$$

$$\approx \boxed{\frac{\pi}{2}}$$