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MATH 231 Final Exam (Review of Exam 1) – Sample Test– Solutions

Problem 1

a) To find S, we need a center and a radius.
Center: The center is given as $(3, 1,8)$
Radius: To find the radius use the distance formulas between center and the given point:

$$d = \sqrt{(4-3)^2 + (1+1)^2 + (8-6)^2} = \sqrt{1+4+4} = 3$$

So the overall equation of the sphere is:

$$(x-3)^2 + (y-1)^2 + (z-8)^2 = 9$$

b) The xy-plane is where $z = 0$, so to find the curve of intersection, set $z = 0$ in your answer to part (a).

$$(x-3)^2 + (y-1)^2 + (0-8)^2 = 9$$

$$(x-3)^2 + (y-1)^2 = -55$$

Problem 2

a) To make vectors parallel, they must be scalar multiples of each other.

$$v_1 = \langle 2, -3, -x \rangle$$
$$v_2 = \langle x, 6, -8 \rangle$$

Looking at the middle terms, you scan see that the second vector is -2 times the first vector. That gives us…

$$(2)(-2) = \text{ and } (-x)(-2) = -8$$

Solving either of these equations gives us

$$x = -4$$

Note: Alternatively you could have taken the cross product of the two vectors and set that equal to 0 to solve for x.

b) To make vectors orthogonal, their dot product must equal 0.

$$\langle 2, -3, -x \rangle \cdot \langle x, 6, -8 \rangle = 2x - 18 + 8x$$

$$10x - 18 = 0$$

$$x = 9/5$$

Problem 3

a) To find a vector perpendicular to a plane, you need to take the cross product of two vectors within the plane. To construct two vectors within the plane find $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$$\overrightarrow{PQ} = \langle 4-0, 1+2, -2-0 \rangle = \langle 4, 3, -2 \rangle$$

$$\overrightarrow{PR} = \langle 5-0, 3+2, 1-0 \rangle = \langle 5, 5, 1 \rangle$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & -2 \\ 5 & 5 & 1 \end{vmatrix} = 13\mathbf{i} - 14\mathbf{j} + 5\mathbf{k}$$

So the vector orthogonal to the plane is:

$$\langle 13, -14, 5 \rangle$$

b) The area of triangle PQR is:

$$A_{\triangle PQR} = \frac{1}{2}\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\|$$

$$A_{\triangle PQR} = \frac{1}{2}\sqrt{(13)^2 + (-14)^2 + (5)^2}$$

$$A_{\triangle PQR} = \frac{1}{2}\sqrt{390}$$

Problem 4

To find the volume of a parallelepiped use the scalar triple product. First you must use the points given to construct 3 vectors:

$$\overrightarrow{PQ} = \langle -1-3, 2-0, 5-1, \rangle = \langle -4, 2, 4 \rangle$$

$$\overrightarrow{PR} = \langle 5-3, 1-0, -1-1, \rangle = \langle 2, 1, -2 \rangle$$

$$\overrightarrow{PS} = \langle 0-3, 4-0, 2-1, \rangle = \langle -3, 4, 1 \rangle$$

$$\begin{vmatrix} -4 & 2 & 4 \\ 2 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix} = -4(1+8) - 2(2-6) + 4(8+3)$$

$$\text{Volume} = 16$$

Problem 5
Vectors are coplanar if their triple product equals 0.

$$\begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 1(-18+36) - 4(36-0) - 7(-18-0) = 18 - 144 + 126 = 0$$

Since the triple product equals 0, the vectors are coplanar.

Problem 6
Start by determining if the two lines are parallel by looking at their direction vectors:

$$\overrightarrow{\mathbf{v}_{\mathbf{L_1}}} = \langle 4, 3, -1 \rangle$$
$$\overrightarrow{\mathbf{v}_{\mathbf{L_2}}} = \langle 2, 1, 4 \rangle$$

Since these vectors are NOT SCALAR MULTIPLES, they are NOT PARALLEL LINES.

Next, see if they are intersecting by setting each the $x, y,$ and z of L_1 equal to the corresponding $x, y,$ and z of L_2. This will create a system of equations. See if there is a solution of s and t that solves this system...

$$\begin{cases} 4 + 4t = 2s \\ -2 + 3t = 3 + s \\ 4 - t = -3 + 4s \end{cases}$$

Solving the first two equations gives us:

$$t = 7, s = s = 16$$

Now, plug in $t = 7, s = s = 16$ in to the third equation above. If it works in the third equation, then the lines intersect. If it does not work, then the lines are skew...

$$4 - 7 = -3 + 4(16)$$

This is not true, so the lines are NOT INTERSECTING.

Therefore, the **LINES ARE SKEW.**

Problem 7

a) To find the equation of any plane you need a point and a normal vector.

Point: Choose any point. I'll use $P(2,2,0)$

Normal Vector: To find the normal vector cross any two vectors on the plane. Make sure to realize that you are given points, not vectors, so you'll first need to construct vectors from those points.

$$\overrightarrow{PQ} = \langle 3-2, 3-2, 1-0, \rangle = \langle 1,1,1 \rangle$$

$$\overrightarrow{PR} = \langle 0-2, -3-2, -3-0 \rangle = \langle -2,-5,-3 \rangle$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ -2 & -5 & -3 \end{vmatrix} = 2\mathbf{i} + 1\mathbf{j} - 3\mathbf{k}$$

So the normal vector is: $\langle 2,1,-3 \rangle$

Now that we have a point and a normal vector, plug into the equation of a plane:

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$2(x-2) + 1(y-2) - 3(z-0) = 0$$

$$2x - 4 + y - 2 - 3z = 0$$

$$2x + y - 3z = 6$$

b) To find the equation of a line you need a point and a direction vector of the line.

Point: Given as $(1,2,-1)$
Direction Vector: This is the normal vector of the plane that we found in part (a): $\langle 2,1,-3 \rangle$

Parametric Equation of the Line:

$$x = 1 + 2t$$
$$y = -2 + t$$
$$z = 1 - 3t$$

Problem 8

To find an equation of any plane you need a point on the plane and a normal vector of the plane.

Point: $(3, -4, 5)$

Normal Vector:

To find the normal vector, find the cross product of any two vectors on the planes.

The first vector on the plane can be obtained from the given line:

$$\overrightarrow{\mathbf{v_1}} = < -2, 3, 5 >$$

The second vector can be obtained by finding the vector between any two points on the plane. We are already given one point on the plane as $(3, -4, 5)$. To find another point on the plane, use the given line.

$$x = -2 - 2t$$
$$y = -1 + 3t$$
$$z = 3 + 5t$$

We can find a point on the line, by choosing any value for t and solving for $x, y,$ and z. I will let $t = 0$, which yields the point $(-2, -1, 3)$.

Now that we have two points we can construct our second vector as:

$$\overrightarrow{\mathbf{v_2}} = < 3 - (-2), -4 - (-1), 5 - 3 > = < 5, -3, 2 >$$

Since we have two vectors on the plane, now we can cross them to get the normal vector of the plane:

$$\overrightarrow{\mathbf{v_1}} \times \overrightarrow{\mathbf{v_2}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 5 \\ 5 & -3 & 2 \end{vmatrix} = \mathbf{i}(6 + 15) - \mathbf{j}(-4 - 25) + \mathbf{k}(6 - 15)$$

Normal Vector: $\vec{\mathbf{n}} = < 21,\ 29, -9 >$

Equation of the Plane:

$$21(x - 3) + 29(y + 4) - 9(z - 5) = 0$$

$$21x - 63 + 29y + 116 - 9z + 45 = 0$$

$$21x + 29y - 9z = -98$$

Note: There are multiple correct answers for the equation of the plane, depending on what value of t you use to obtain your point on the line, and what order you subtract your points in order to get your second vector.

Problem 9
To determine if planes are parallel, look at their normal vectors.

$$\overrightarrow{\mathbf{n_p}} = \langle 1, 1, 3 \rangle$$
$$\overrightarrow{\mathbf{n_q}} = \langle 2, 1, -1 \rangle$$

These vectors are **NOT SCALAR MULTIPLES** of each other, so the planes are **NOT PARALLEL.** Therefore they are **INTERSECTING.**

Line of Intersection
For an equation of any line you need a point and a direction vector:

Point: To find a point, set x or y or $z = 0$. In this example, I will set $z = 0$. Setting $z = 0$ and plugging into the two equations of the planes gives us the following system of equations:

$$\begin{cases} x + y = 5 \\ 2x + y = 4 \end{cases}$$

Solving this system of equations gives us $x = -1, y = 6$.

So the point is: $(-1, 6, 0)$

(Note that there are infinitely many points on a line, so if you chose to set x or $y = 0$ instead, you may end up with a different point and that is okay!)

Direction Vector: To find the direction vector of the line, you need to take the cross product of the two normal vectors of the planes.

$$\overrightarrow{\mathbf{n_p}} \times \overrightarrow{\mathbf{n_q}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 3 \\ 2 & 1 & -1 \end{vmatrix} = -4\mathbf{i} + 7\mathbf{j} - 1\mathbf{k}$$

So the direction vector of the line is: $\langle -4, 7, -1 \rangle$

The parametric equation of the line is:

$$x = -1 - 4t$$
$$y = 6 + 7t$$
$$z = -t$$

Angle Between the Planes
To find the angle between the planes use the formula:

$$\cos\theta = \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|}$$

$$\cos\theta = \frac{\langle 1, 1, 3 \rangle \cdot \langle 2, 1, -1 \rangle}{|\mathbf{n_1}||\mathbf{n_2}|} = 0$$

$$\cos\theta = 0$$

$$\theta = \frac{\pi}{2}$$

Problem 10

Equation	Picture	Name of Surface
$x^2 + y^2 = 1$	C	Cylinder
$2x^2 - 2y^2 - 2z = 0$	A	Hyperbolic Paraboloid
$z^2 - 9x^2 - 4y^2 = 0$	D	Cone
$z = 4x^2 + y^2$	B	Elliptic Paraboloid

Problem 11
Use the second equation to make a polar parameterization. This gives you:

$$x = 2\cos t$$

$$z = 2\sin t$$

Then use the first equation to solve for y, and plug in the values of x and z above:

$$y = 1 + 2z - x$$

$$y = 1 + 4\sin t - 2\cos t$$

So the overall vector function is:

$$\mathbf{r}(t) = \langle 2\cos t\,, 1 + 4\sin t - 2\cos t\,, 2\sin t \rangle$$

Problem 12

a) Since one variable, y, is already solved in terms of another variable, z, we will use a trivial parameterization by letting $z = t$.

   This gives us:

$$x = 5 - t - 2t^2$$
$$y = 2t^2$$
$$z = t$$

   So $\mathbf{r}(t) = \langle 5 - t - t^2,\ 2t^2,\ t \rangle$

b) To find the equation of any line we need a point and direction vector.

   Point: $(-5,\ 8,\ 2)$

   Direction Vector: Find this by taking $\mathbf{r}'(t)$:

$$\mathbf{r}'(t) = \langle -1 - 4t, 4t, 1 \rangle$$

   We want to evaluate the direction vector at $(-1, 8, 2)$ which corresponds to $t = 2$.

$$\mathbf{r}'(2) = \langle -1 - 4(2), 4(2), 1 \rangle = \langle -9, 8, 1 \rangle$$

   The equation of the line is:

$$x = -5 - 9t$$
$$y = 8 + 8t$$
$$z = 2 + t$$

Problem 13

Recall that for any line you need a point and a direction vector.

Point: $\left(1, \sqrt{3}, 2\right)$

Direction Vector: Find $\mathbf{r}'(t)$

$$\mathbf{r}'(t) = \langle 2\cos t\,, 4\cos(2t)\,, 6\cos(3t)\rangle$$

To find the t values that corresponds to given point, set $2\sin t = 1$ (or $2\sin(2t) = \sqrt{3}$, or $2\sin(3t) = 2$)

This gives us:

$$2\sin t = \frac{1}{2}$$

$$t = \pi/6$$

$$\mathbf{r}'(\pi/6) = \langle\sqrt{3}, 2{,}0\rangle$$

So the parametric equation of the line is:

$$x = 1 + \sqrt{3}t$$
$$y = \sqrt{3} + 2t$$
$$z = 2$$

Problem 14

To calculate arc length first find $\mathbf{r}'(t)$.

$$r'(t) = \langle 5, -2\sin(2t), 2\cos(2t)\rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{25 + 4\sin^2(2t) + 4\cos^2(2t)} = \sqrt{29}$$

$$\int_0^3 \sqrt{29}\, dt = \sqrt{29}t = 3\sqrt{29}$$

Problem 15

$$\mathbf{v}(t) = \int (-\cos t\,\mathbf{i} + e^{2t}\mathbf{j} - 5\mathbf{k})\, dt$$

$$\mathbf{v}(t) = -\sin t\,\mathbf{i} + \frac{1}{2}e^{2t}\mathbf{j} - 5t\mathbf{k} + C$$

To find C, use $\mathbf{v}(0) = \langle 0,3,1\rangle = 3\mathbf{j} + \mathbf{k}$

$$3\mathbf{j} + \mathbf{k} = -\sin 0\,\mathbf{i} + \frac{1}{2}e^{2(0)}\mathbf{j} - 5(0)\mathbf{k} + C$$

$$3\mathbf{j} + \mathbf{k} = \frac{1}{2}\mathbf{j} + C$$

$$C = \frac{5}{2}\mathbf{j} + \mathbf{k}$$

$$\mathbf{v}(t) = (-\sin t)\mathbf{i} + \left(\frac{1}{2}e^{2t} + \frac{5}{2}\right)\mathbf{j} + (-5t + 1)\mathbf{k}$$

$$\mathbf{r}(t) = \int \left[(-\sin t)\mathbf{i} + \left(\frac{1}{2}e^{2t} + \frac{5}{2}\right)\mathbf{j} + (-5t + 1)\mathbf{k}\right] dt$$

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \left(\frac{1}{4}e^{2t} + \frac{5}{2}t\right)\mathbf{j} + \left(-\frac{5}{2}t^2 + t\right)\mathbf{k} + C$$

To find C, use $\mathbf{r}(0) = \langle -1,3,2\rangle = -\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

$$-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} = \cos(0)\,\mathbf{i} + \left(\frac{1}{4}e^{2(0)} + \frac{5}{2}(0)\right)\mathbf{j} + \left(-\frac{5}{2}(0)^2 + (0)\right)\mathbf{k} + C$$

$$-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} = \mathbf{i} + \frac{1}{4}\mathbf{j} + C$$

$$C = -2\mathbf{i} + \frac{11}{4}\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{r}(t) = (\cos t - 2)\mathbf{i} + \left(\frac{1}{4}e^{2t} + \frac{5}{2}t + \frac{11}{4}\right)\mathbf{j} + \left(-\frac{5}{2}t^2 + t + 2\right)\mathbf{k} + C$$

Problem 16

a)

$$\mathbf{r}(t) = \int (2\mathbf{i} + 2e^{2t}\mathbf{j} + 4t\mathbf{k})\, dt$$

$$\mathbf{r}(t) = 2t\mathbf{i} + e^{2t}\mathbf{j} + 2t^2\mathbf{k} + C$$

Use $\mathbf{r}(0) = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, to plug in and solve for C…

$$\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} = 2(0)\mathbf{i} + e^{2(0)}\mathbf{j} + 2(0)^2\mathbf{k} + C$$

$$\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} = \mathbf{j} + C$$

$$\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} = C$$

$$\mathbf{r}(t) = (2t + 1)\mathbf{i} + (e^{2t} + 2)\mathbf{j} + (2t^2 + 2)\mathbf{k} + C$$

b) First, find when speed is $\sqrt{20 + 4e^4}$ by setting the magnitude of the velocity equation to that value.

$$\|\mathbf{v}(t)\| = \sqrt{20 + 4e^4}$$

$$\sqrt{(2)^2 + (2e^{2t})^2 + (4t)^2} = \sqrt{20 + 4e^4}$$

$$4 + 4e^{4t} + 16t^2 = 20 + 4e^4$$

$$20t^2 + 4e^{4t} = 20 + 4e^4$$

$$t = 1$$

So this means, we want to find the acceleration at $t = 1$

$$\mathbf{v}(t) = 2\mathbf{i} + 2e^{2t}\mathbf{j} + 4t\mathbf{k}$$

$$\mathbf{a}(t) = 0\mathbf{i} + 4e^{2t}\mathbf{j} + 4\mathbf{k}$$