MATH 250 Exam 2 - Sample Test Solutions

Problem 1

$$y'' + 9y = 5\sin(2x)$$

Find y_c:

$$r^2 + 9 = 0$$
$$r^2 = -9$$
$$r = \pm 3i$$

$$y_c = c_1\cos(3x) + c_2\sin(3x)$$

Find y_p:

$$y_p = A\cos(2x) + B\sin(2x)$$
$$y_p' = -2A\sin(2x) + 2B\cos(2x)$$
$$y_p'' = -4A\cos(2x) - 4B\sin(2x)$$

$$y'' + 9y = 5\sin(2x)$$

$$-4A\cos(2x) - 4B\sin(2x) + 9A\cos(2x) + 9B\sin(2x) = 5\sin(2x)$$

$$5A\cos(2x) + 5B\sin(2x) = 0\cos(2x) + 5\sin(2x)$$

$$A = 0, \quad 5B = 5$$
$$B = 1$$

$$y_p = \sin(2x)$$

$$y = y_c + y_p$$

$$y = c_1\cos(3x) + c_2\sin(3x) + \sin(2x)$$

$$y(0) = 0 \rightarrow 0 = c_1$$

$$y = c_2\sin(3x) + \sin(2x)$$

$$y' = 3c_2\cos(3x) + 2\cos(2x)$$

$$y'(0) = 8 \rightarrow 8 = 3c_2 + 2$$
$$6 = 3c_2$$
$$c_2 = 2$$

$$y = 2\sin(3x) + \sin(2x)$$

Problem 2

$$y'' + 25y = 3\cos(5x)$$

Find y_c:

$r^2 + 25 = 0$

$r = \pm 5i$

$$y_c = C_1\cos(5x) + C_2\sin(5x)$$

Find y_p:

$y_p = A\cos(5x) + B\sin(5x)$ → need to modify

$$y_p = x(A\cos(5x) + B\sin(5x))$$

$$y_p' = x(-5A\sin(5x) + 5B\cos(5x)) + A\cos(5x) + B\sin(5x)$$

$$y_p'' = x(-25A\cos(5x) - 25B\sin(5x)) - 5A\sin(5x) + 5B\cos(5x) - 5A\sin(5x) + 5B\cos(5x)$$

$$= -25Ax\cos(5x) - 25Bx\sin(5x) - 10A\sin(5x) + 10B\cos(5x)$$

$$y'' + 25y = 3\cos(5x)$$

$$-25Ax\cos(5x) - 25Bx\sin(5x) - 10A\sin(5x) + 10B\cos(5x) + 25Ax\cos(5x) + 25Bx\sin(5x) = 3\cos(5x)$$

$$-10A\sin(5x) + 10B\cos(5x) = 3\cos(5x)$$

$$A = 0, \quad B = \frac{3}{10}$$

$$y_p = \frac{3}{10}x\sin(5x)$$

$$y = C_1\cos(5x) + C_2\sin(5x) + \frac{3}{10}x\sin(5x)$$

Problem 3

a) True

b) False $A = \sqrt{C_1^2 + C_2^2}$

$A = \sqrt{(2)^2 + (2)^2} = \sqrt{8}$

c) False. It will remain constant

Problem 4

a) $x'' + 4x' + 9x = 0$

$r^2 + 4r + 9 = 0$

$r = \frac{-4 \pm \sqrt{16 - 4(1)(9)}}{2} = \frac{-4 \pm \sqrt{-20}}{2} = \frac{-4 \pm 2\sqrt{5}i}{2} = -2 \pm \sqrt{5}i$

$x(t) = C_1 e^{-2t} \cos(\sqrt{5}t) + C_2 e^{-2t} \sin(\sqrt{5}t)$

$x(0) = 0 \rightarrow \quad 0 = C_1 + 0$

$C_1 = 0$

$x(t) = C_2 e^{-2t} \sin(\sqrt{5}t)$

$x'(t) = C_2 e^{-2t} \cdot \sqrt{5}\cos(\sqrt{5}t) - 2C_2 e^{-2t} \sin(\sqrt{5}t)$

$x'(0) = 1 \rightarrow \quad 1 = \sqrt{5}C_2 - 0$

$C_2 = \frac{1}{\sqrt{5}}$

$x(t) = \frac{1}{\sqrt{5}} e^{-2t} \sin(\sqrt{5}t)$

b) underdamped

c) $x(t) = 0$

$\frac{1}{\sqrt{5}} e^{-2t} \sin(\sqrt{5}t) = 0$

$\sin(\sqrt{5}t) = 0$

$\sqrt{5}t = 0 + 2\pi n$

$t = \frac{2\pi n}{\sqrt{5}}$

Problem 5

$m = 1$

$F_s = 10, \ x = \frac{5}{2} \quad \rightarrow \quad F_s = kx$

$10 = k\frac{5}{2}$

$c = 0$

$k = 4$

a) $mx'' + cx' + kx = 0$

$x'' + 4x = 0, \quad x(0) = 1, \quad x'(0) = 0$

b) $x'' + 4x = 0$

$r^2 + 4 = 0$

$r^2 = -4$

$r = 0 \pm 2i$

$x(t) = C_1 \cos(2t) + C_2 \sin(2t)$

$x(0) = 1 \rightarrow 1 = C_1 + 0$

$x(t) = \cos(2t) + C_2 \sin(2t)$

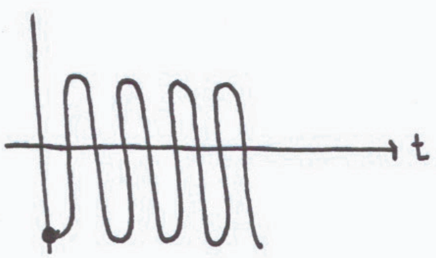
$x'(t) = -2\sin(2t) + 2C_2\cos(2t)$

$x'(0) = 0 \rightarrow 0 = 0 + 2C_2$

$C_2 = 0$

$x(t) = \cos(2t)$

c)



lowest = 1m below eq.

d) $x(t) = 0$

$\cos(2t) = 0$

$2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

$t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4} \ldots$

time between $= \frac{\pi}{2}$ s

Problem 6

a) $x(t) = 0$

$$3e^{-4t} - 5e^{-3t} = 0$$

$$3e^{-4t} = 5e^{-3t}$$

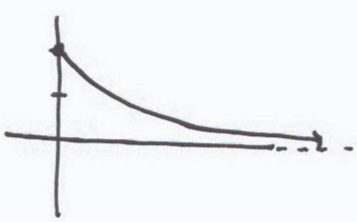
$$\frac{3}{5} = \frac{e^{-3t}}{e^{-4t}}$$

$\frac{3}{5} = e^{t} \rightarrow t = \ln\left(\frac{3}{5}\right)$, but this is a negative number so it does **NOT** pass through eq.

b+c) Initial Pos.: $x(0) = 3e^{0} - 5e^{0} = -2 \rightarrow$ above eq.

$$x'(t) = -12e^{-4t} + 15e^{-3t}$$

Initial Vel.: $x'(0) = -12 + 15 = 3 \rightarrow$ downward



most stretched: as $t \rightarrow \infty$

most compressed: at $t = 0$

Problem 7

$m = 2$

$F_s = 16, \; x = 2$

$c = 0$

$$mx'' + cx' + kx = 9\cos(\omega t)$$

Find k:

$F_s = kx$

$16 = k \cdot 2$

$k = 8$

$$2x'' + 8x = 9\cos(\omega t)$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{2}} = 2$$

$$\boxed{\omega = 2}$$

Problem 8

$m = 1$

$c = 0$

$k = 9$

$x(0) = 0$

$x'(0) = 0$

a) $x'' + 9x = 10\cos(3t), \quad x(0) = 0, \; x'(0) = 0$

b) Find x_c:

$r^2 + 9 = 0$

$r = \pm 3i$

$x_c = c_1\cos(3t) + c_2\sin(3t)$

Find x_p:

$x_p = A\cos(3t) + B\sin(3t)$ → need to modify

$x_p = t(A\cos(3t) + B\sin(3t))$

$x_p' = t(-3A\sin(3t) + 3B\cos(3t)) + A\cos(3t) + B\sin(3t)$

$x_p'' = t(-9A\cos(3t) - 9B\sin(3t)) - 3A\sin(3t) + 3B\cos(3t) - 3A\sin(3t) + 3B\cos(3t)$

$= -9At\cos(3t) - 9Bt\sin(3t) - 6A\sin(3t) + 6B\cos(3t)$

$x'' + 9x = 10\cos(3t)$

$9At\cos(3t) - 9Bt\sin(3t) - 6A\sin(3t) + 6B\cos(3t) + 9At\cos(3t) + 9Bt\sin(3t) =$

$10\cos(3t)$

$-6A\sin(3t) + 6B\cos(3t) = 0\sin(3t) + 10\cos(3t)$

$A = 0, \quad 6B = 10$

$B = \frac{10}{6} = \frac{5}{3}$

$x_p = \frac{5}{3}t\sin(3t)$

$x(t) = x_c + x_p$

$x(t) = c_1\cos(3t) + c_2\sin(3t) + \frac{5t}{3}\sin(3t)$

$$x(0)=0 \rightarrow 0 = C_1 + 0 + 0$$
$$C_1 = 0$$

$$x(t) = C_2 \sin(3t) + \frac{5}{3}t \sin(3t)$$

$$x'(t) = 3C_2 \cos(3t) + \frac{5}{3}t(3\cos(3t)) + \sin(3t)\cdot\frac{5}{3}$$

$$x'(0)=0 \rightarrow 0 = 3C_2 + 0 + 0$$
$$C_2 = 0$$

$$x(t) = \frac{5}{3}t \sin(3t)$$

c) yes

Problem 9

a) $A - 4I$

$$\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 5 \\ -2 & 0 \end{bmatrix}$$

b) $A\vec{v}$

$$\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} (3)(6) + (5)(1) \\ (-2)(6) + (4)(1) \end{bmatrix} = \begin{bmatrix} 23 \\ -8 \end{bmatrix}$$

c) $A\vec{x}$

$$\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x + 5y \\ -2x + 4y \end{bmatrix}$$

d) $$\det\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} = (3)(4) - (-2)(5) = 12 + 10 = 22$$

Problem 10

$$x_1' = 1x_1 + 3x_2$$

$$x_2' = -1x_1 + 0x_2$$

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$