Problem 1

a) $h(90,7) = 15$, When the plant is given 90 mL of water, and 7 mg of fertilizer, its height is 15 inches.

$h_f(90,7) = -0.5$, When the plant is given 90 mL of water, and 7 mg of fertilizer, its height is decreasing at a rate 0.5 inches/mg of fertilizer.

$h_w(90,7) = 1.2$, When the plant is given 90 mL of water, and 7 mg of fertilizer, its height is increasing at a rate of 1.2 inches/mL of water.

b) $h(88,8) \approx h(90,7) + 1.2(88 - 90) - 0.5(8 - 7)$

$h(88,8) \approx 15 + 1.2(-2) - 0.5(1) = 15 - 2.4 - 0.5 = 12.1$ inches

Problem 2

Recall the formula for the equation of the tangent plane is given by: $z - c = f_x(a,b)(x-a) + f_y(a,b)(y-b)$

First find the point:

$$z = x^2y^2 + x - y + 2$$

$$@\ x=1,\ y=0 \rightarrow z = (1)^2(0)^2 + 1 - 0 + 2 \rightarrow z = 3$$

Point: $(\underset{a}{1}, \underset{b}{0}, \underset{c}{3})$

Now find $f_x(a,b)$ and $f_y(a,b)$:

$$f_x(a,b) = 2xy^2 + 1 \rightarrow f_x(1,0) = 1$$

$$f_y(a,b) = 2x^2y - 1 \rightarrow f_y(1,0) = -1$$

The equation of the tangent line is:

$$z - 3 = 1(x-1) + -1(y-0)$$

$$z - 3 = x - 1 - y$$

$$\boxed{x - y - z = -2}$$

Problem 3

Recall that the linear approximation formula is given by: $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(y - b)$

$$f_x = \frac{-x}{\sqrt{9-x^2-4y^2}} \longrightarrow f_x(2,1) = \frac{-2}{\sqrt{9-4-4}} = \frac{-2}{1} = -2$$

$$f_y = \frac{-4y}{\sqrt{9-x^2-4y^2}} \longrightarrow f_y(2,1) = \frac{-4}{\sqrt{9-4-4}} = \frac{-4}{1} = -4$$

$$f(2,1) = \sqrt{9-(2)^2-4(1)^2} = 1$$

$$L(x,y) = 1 - 2(x-2) - 4(y-1)$$

$$L(1.98, 1.02) = 1 - 2(1.98-2) - 4(1.02-1)$$

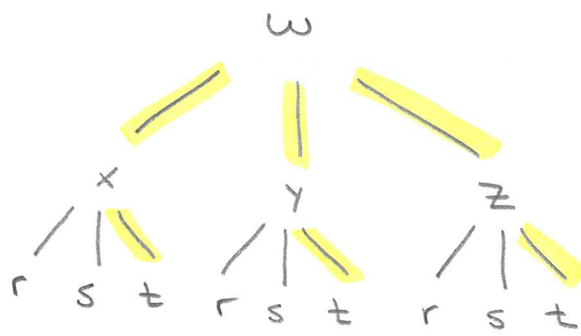
$$= 1 - 2(-.02) - 4(.02)$$

$$= 1 + .04 - .08$$

$$\boxed{= .96}$$

Problem 4

Use the chain rule. Remember it is helpful to draw a tree diagram before you start:



$$\frac{\partial w}{\partial t} = \frac{dw}{dx}\frac{\partial x}{\partial t} + \frac{dw}{dy}\frac{\partial y}{\partial t} + \frac{dw}{dz}\cdot\frac{\partial z}{\partial t}$$

$$\frac{\partial w}{\partial t} = \left(3x^2y - \frac{z}{x^2}\right)\left(-\sin(rst)(rs) + s^2e^{s^2t}\right) + (x^3)(r^2s) + \left(\frac{1}{x}\right)(\ln s)$$

$@\ (r, s, t) = (\pi, 3, 0)$

$x = \cos(0) + e^0 = 2$

$y = 0$

$z = 0$

$$\frac{\partial w}{\partial t} = (0) + (2)^3(\pi^2)(3) + \left(\frac{1}{2}\right)\ln(3)$$

$$\boxed{\frac{\partial w}{\partial t} = 24\pi^2 + \frac{1}{2}\ln(3)}$$

Problem 5

a) To find the rate of change of $f(x,y)$ in the direction of a vector…

1. Find $\nabla f(x,y)$:

$f_x = -y^2e^x \rightarrow f_x(0,1) = -1$

$f_y = y(-xe^{xy}) + e^{-xy} \rightarrow f_y(0,1) = 0 + e^0 = 1$

$\nabla f(x,y) = \langle -1, 1 \rangle$

2. Find $\vec{u}$:

$\vec{u} = \dfrac{\langle 3, 4 \rangle}{\sqrt{9+16}} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

3. Take the dot product of $\nabla f(x,y)$ and $\vec{u}$:

$\langle -1, 1 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle$

$= -\frac{3}{5} + \frac{4}{5}$

$\boxed{= \frac{1}{5}}$

b) The direction of steepest ascent is $\nabla f(x,y)$ at (0,1):

$\boxed{\nabla f(0,1) = \langle -1, 1 \rangle}$

c) The maximum rate of change of $f(x,y)$ is at the magnitude of $\nabla f(x,y)$ at (0,1):

$|\langle -1, 1 \rangle| = \sqrt{(-1)^2 + (1)^2} = \boxed{\sqrt{2}}$

Problem 6

a) Directional Derivative

$$\nabla P = \left\langle \frac{2xz}{y}, \frac{-x^2 z}{y^2}, \frac{x^2}{y} \right\rangle$$

$$\nabla P(2,2,1) = \left\langle \frac{4}{2}, \frac{-4}{4}, \frac{4}{2} \right\rangle = \langle 2, -1, 2 \rangle$$

$$\vec{v} = \langle 3-2, 1-2, 0-1 \rangle = \langle 1, -1, -1 \rangle$$

$$\vec{u} = \frac{\langle 1, -1, -1 \rangle}{\sqrt{1+1+1}} = \left\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle$$

$$D_u = \langle 2, -1, 2 \rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle = \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}} = \boxed{\frac{1}{\sqrt{3}}}$$

b) Negative Gradient

$$-\nabla P(2,2,1) = \boxed{\langle -2, 1, -2 \rangle}$$

Problem 7

$$\underbrace{x^2 + y^2 + z^2 - 5e^{xy}}_{F(x,y,z)} = 4 \rightarrow \text{Level Curve}$$

point: $(1,0,2)$

$\vec{n} = \nabla F : \langle 2, -5, 4 \rangle$

$\hookrightarrow F_x = 2x - 5ye^{xy} \rightarrow F_x(1,0,2) = 2$

$F_y = 2y - 5xe^{xy} \rightarrow F_y(1,0,2) = 0 - 5 = -5$

$F_z = 2z \rightarrow F_z(1,0,2) = 4$

Eq. of Plane:

$$2(x-1) - 5(y-0) + 4(z-2) = 0$$

$$2x - 2 - 5y + 4z - 8 = 0$$

$$\boxed{2x - 5y + 4z = 10}$$

Problem 8

Start by taking f_x and f_y and setting them equal to 0:

$$f_x = 3x^2 - 6x - 9 = 3(x^2-2x-3)$$
$$f_y = 3y^2 - 6y = 3y(y-2)$$
$$\longrightarrow \begin{cases} 3(x-3)(x+1) = 0 \\ 3y(y-2) = 0 \end{cases}$$

$$\begin{cases} x-3=0 \\ y=0 \end{cases} \quad \begin{cases} x+1=0 \\ y=0 \end{cases} \quad \begin{cases} x-3=0 \\ y-2=0 \end{cases} \quad \begin{cases} x+1=0 \\ y-2=0 \end{cases}$$

$(3,0)$ $\quad$ $(-1,0)$ $\quad$ $(3,2)$ $\quad$ $(-1,2)$

To classify the critical points find $D = f_{xx}(a,b)f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$ and evaluate it at each critical point:

$$f_x = 3x^2 - 6x - 9$$
$$f_{xx} = 6x - 6$$
$$f_{xy} = 0$$
$$f_y = 3y^2 - 6y$$
$$f_{yy} = 6y - 6$$

$$D = (6x-6)(6y-6) - 0$$

@ $(3,0) \rightarrow D = (12)(-6) < 0$

saddle point

@ $(-1,0) \rightarrow D = (-12)(-6) > 0$, and $f_{xx} < 0$

local maximum

@ $(3,2) \rightarrow (12)(6) > 0$, and $f_{xx} > 0$

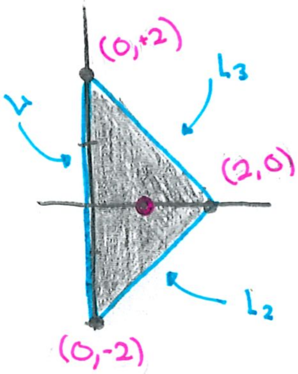
local minimum

@ $(-1,2) \rightarrow (-12)(6) < 0$

saddle point

Problem 9

Step 1: Draw the region



Step 2: Find the critical points of $f(x, y)$:

$f_x = 2x - 2$
$f_y = 2y$
$\rightarrow \begin{cases} 2x - 2 = 0 & \rightarrow x = 1 \\ 2y = 0 & \rightarrow y = 0 \end{cases}$

C.P. : $(1, 0)$

Step 3: Boundary Lines

L_1 : $x = 0$

$f(0, y) = y^2$
$f' = 2y$
$0 = 2y$
$y = 0$
C.P. : $(0, 0)$

L_2: $y = x - 2$

$f(x, x-2) = x^2 + (x-2)^2 - 2x$
$f' = 2x + 2(x-2) - 2$
$0 = 2x + 2x - 4 - 2$
$0 = 4x - 6$
$x = \frac{3}{2}$, $y = \frac{3}{2} - 2 = -\frac{1}{2}$
C.P. : $\left(\frac{3}{2}, -\frac{1}{2}\right)$

L_3. $y = 2 - x$

$f(x, 2-x) = x^2 + (2-x)^2 - 2x$
$f' = 2x + 2(2-x)(-1) - 2$
$0 = 2x - 4 + 2x - 2$
$4x = 6$
$x = \frac{3}{2}$, $y = 2 - \frac{3}{2} = \frac{1}{2}$
C.P. : $\left(\frac{3}{2}, \frac{1}{2}\right)$

Step 4: Evaluate endpoints, overall CP, and boundary CP $\quad f(x, y) = x^2 + y^2 - 2x$

$f(0, 2) = 4 \rightarrow$ Max
$f(0, -2) = 4$
$f(2, 0) = 4 - 4 = 0$
$f(1, 0) = 1 - 2 = -1 \rightarrow$ Min
$f(0, 0) = 0$
$f(3/2, -1/2) = \frac{9}{4} + \frac{1}{4} - 3 = -\frac{1}{2}$
$f(3/2, 1/2) = \frac{9}{4} + \frac{1}{4} - 3 = -\frac{1}{2}$

$$\boxed{\begin{array}{l} \text{Abs Max Value} = 4 \\ \text{Abs Min Value} = -1 \end{array}}$$

Problem 10

Use Lagrange Multipliers:

$d = \sqrt{(x-0)^2 + (y-1)^2 + (z-1)^2}$

$f = x^2 + (y-1)^2 + (z-1)^2$

$g = x + 2y + z$

$\nabla f = \langle 2x, 2(y-1), 2(z-1) \rangle$

$\nabla g = \langle 1, 2, 1 \rangle$

$\nabla f = \lambda \nabla g$

$$\begin{cases} (2x = \lambda)2 \\ 2(y-1) = 2\lambda \\ (2(z-1) = \lambda)2 \\ x + 2y + z = 4 \end{cases} \rightarrow \begin{cases} 4x = 2\lambda \quad (1) \\ 2y - 2 = 2\lambda \quad (2) \\ 4z - 4 = 2\lambda \quad (3) \\ x + 2y + z = 4 \quad (4) \end{cases}$$

(1) = (2)

$4x = 2y - 2$

$2y = 4x + 2$

$y = 2x + 1$

(1) = (3)

$4x = 4z - 4$

$x = z - 1$

$z = x + 1$

Plug into (4):

$x + 2(2x+1) + x + 1 = 4$

$x + 4x + 2 + x + 1 = 4$

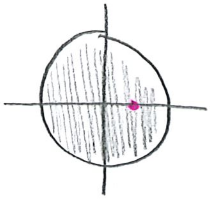
$6x = 1$

$x = \frac{1}{6} \rightarrow y = \frac{4}{3}, \; z = \frac{7}{6}$

$$\boxed{\left(\frac{1}{6}, \frac{4}{3}, \frac{7}{6}\right)}$$

Problem 11

Use Lagrane Multipliers:



1. Find C.P. of $f(x,y)$ to evaluate the inside:

$$f_x = 6x \qquad f_y = 4y-4 \rightarrow \begin{cases} 6x = 0 \rightarrow x = 0 \\ 4y-4=0 \rightarrow y=1 \end{cases}$$

C.P.: $(0,1)$

2. Use Lagrange to evaluate the boundary:

$f = 3x^2+2y^2-4y$ $\qquad$ $g = x^2+y^2$

$\nabla f = \langle 6x, 4y-4 \rangle$ $\qquad$ $\nabla g = \langle 2x, 2y \rangle$

$$\nabla f = \lambda \nabla g$$

$$\begin{cases} (6x = \lambda 2x)y \\ (4y-4 = \lambda 2y)x \\ x^2+y^2=9 \end{cases} \rightarrow \begin{cases} 6xy = \lambda 2xy \quad ① \\ x(4y-4) = \lambda 2xy \quad ② \\ x^2+y^2=9 \quad ③ \end{cases}$$

① = ②

$6xy = 4xy - 4x$

$2xy + 4x = 0$

$2x(y+2)=0$

$x=0,\ y=-2$

Plug into ③:

$x=0$: $y^2=9$

$y = \pm 3$

CP.: (0,3), (0,-3)

$y=-2$: $x^2+4=9$

$x^2=5$

$x = \pm\sqrt{5}$

C.P.: $(\sqrt{5},-2), (-\sqrt{5},-2)$

$f(x,y) = 3x^2+2y^2-4y$

$f(0,1) = 2-4 = -2$

$f(0,3)=18-12=6$

$f(0,-3)=18+12=30$

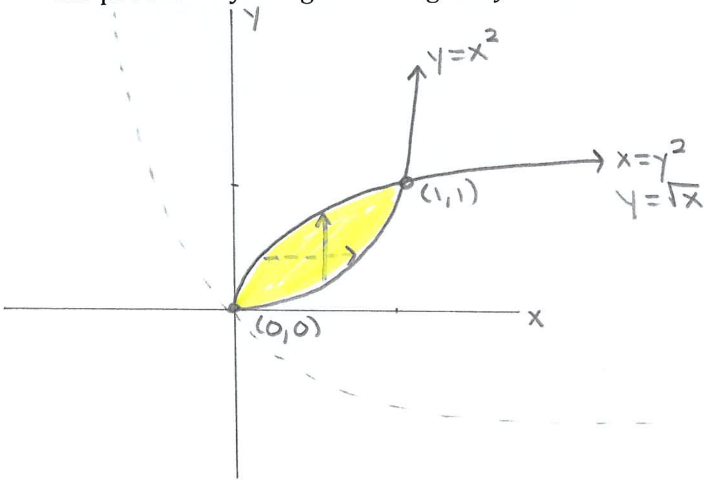
$f(-\sqrt{5},2) = 15+8-8 = 15$

$f(\sqrt{5},2) = 15$

Absolute Min Value: -2 Absolute Max Value: 30

Problem 12

Start by graphing the region. In this particular region, you can choose to integrate in either direction. I will solve the problem by integrate along the y's first.



To find $f(x, y)$, the function that goes inside the integral, solve for z in the given equation:

$$4x + 6y - 2z = 0$$
$$2z = 4x + 6y$$
$$z = 2x + 3y$$

$$\int_{x=0}^{x=1} \int_{y=x^2}^{y=\sqrt{x}} (2x + 3y)\, dy\, dx = \int_{x=0}^{x=1} \cdot \left[2xy + \frac{3}{2}y^2 \right]_{y=x^2}^{y=\sqrt{x}} dx$$

$$\left[2x\sqrt{x} + \frac{3}{2}(\sqrt{x})^2\right] - \left[2xx^2 + \frac{3}{2}(x^2)^2\right]$$
$$= 2x^{3/2} + \frac{3}{2}x - 2x^3 - \frac{3}{2}x^4$$

$$= \int_{x=0}^{x=1} \left(2x^{3/2} + \frac{3}{2}x - 2x^3 - \frac{3}{2}x^4\right) dx$$

$$= \left. \frac{4}{5}x^{5/2} + \frac{3}{4}x^2 - \frac{1}{2}x^4 - \frac{3}{10}x^5 \right]_{x=0}^{x=1}$$

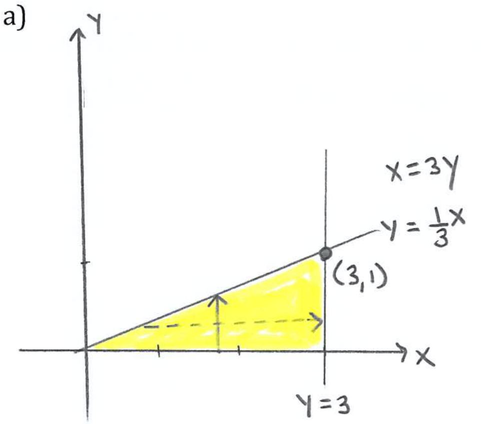
$$= \left[\frac{4}{5} + \frac{3}{4} - \frac{1}{2} - \frac{3}{10}\right] - [0]$$

$$= \frac{16}{20} + \frac{15}{20} - \frac{10}{20} - \frac{6}{20} = 3/4$$

Problem 13

Each of these iterated integrals are impossible to solve in the given order, so we must switch the order of integration. It is helpful to draw the region so find the new bounds of integration:

a)



$$\int_{x=0}^{x=3} \int_{y=0}^{y=\frac{1}{3}x} e^{x^2}\, dy\, dx$$

$$= \int_{x=0}^{x=3} y e^{x^2} \Big]_{y=0}^{y=\frac{1}{3}x} dx \qquad \left[\frac{1}{3}x e^{x^2}\right] - [0]$$

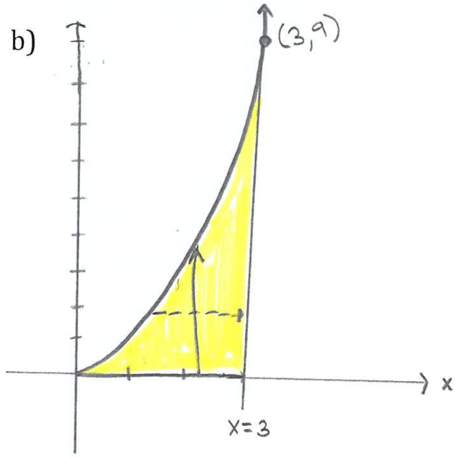
$$= \int_{x=0}^{x=3} \frac{1}{3} x e^{x^2}\, dx \qquad \begin{aligned} u &= x^2 \\ du &= 2x\, dx \\ dx &= \frac{du}{2x} \end{aligned}$$

$$= \int \frac{1}{3} x e^{u} \frac{du}{2x} = \frac{1}{6} \int e^{u}\, du = \frac{1}{6} e^{x^2} \Big]_{x=0}^{x=3}$$

$$= \frac{1}{6} e^{9} - \frac{1}{6} e^{0}$$

$$= \boxed{\frac{e^9 - 1}{6}}$$

b)



$x = \sqrt{y} \rightarrow y = x^2$

$x = 3$

$y = 0$

$y = 9$

$$\int_{x=0}^{x=3} \int_{y=0}^{y=x^2} \sin(x^3)\, dy\, dx$$

$$= \int_{x=0}^{x=3} y\sin(x^3) \Big]_{y=0}^{y=x^2} dx \rightarrow x^2\sin(x^3) - 0$$

$$= \int_{x=0}^{x=3} x^2 \sin(x^3)\, dx$$

$u = x^3$

$du = 3x^2 dx$

$dx = \dfrac{du}{3x^2}$

$$= \int x^2 \sin(u) \frac{du}{3x^2} = \int \frac{1}{3} \sin(u)\, du = -\frac{1}{3}\cos(u)$$

$$= -\frac{1}{3}\cos(x^3) \Big]_{x=0}^{x=9}$$

$$= -\frac{1}{3}\cos(27) + \frac{1}{3}\cos(0)$$

$$= \boxed{\frac{1 - \cos(27)}{3}}$$

Problem 14

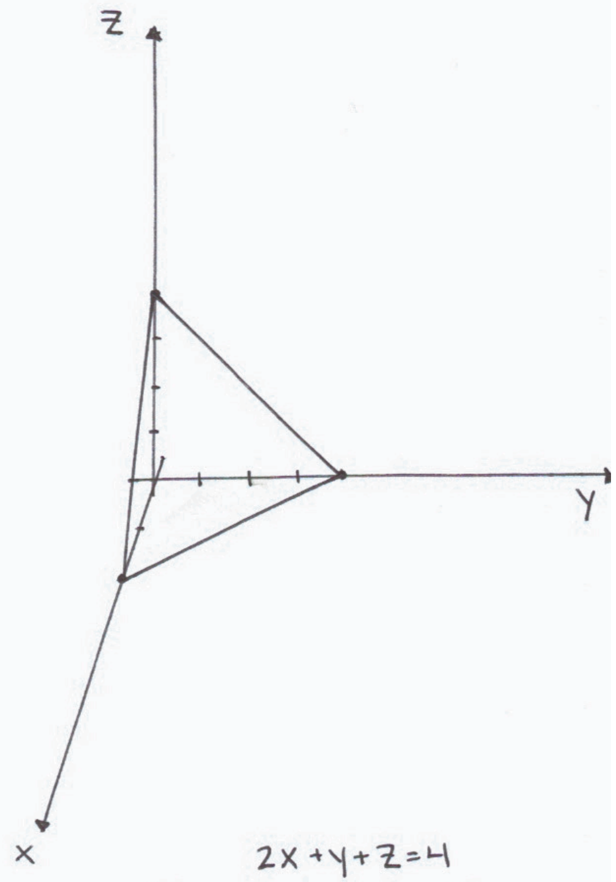
$2x + y + z = 4 \rightarrow z = 4 - 2x - y$

$0 \le z \le 4-2x-y$

$0 \le y \le 4-2x$

$0 \le x \le 2$

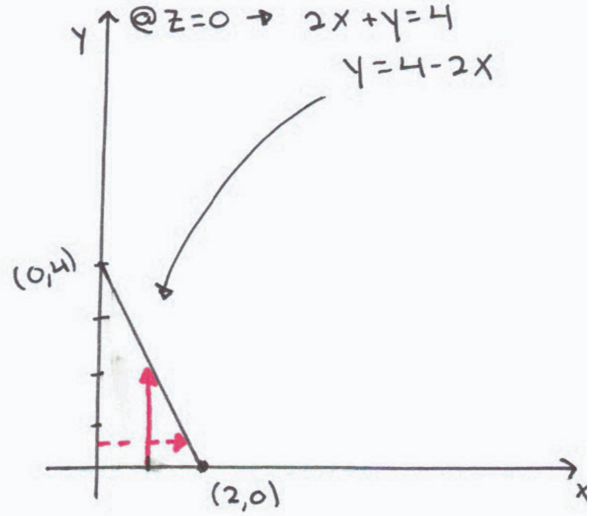
$$\boxed{\int_{x=0}^{x=2} \int_{y=0}^{y=4-2x} \int_{z=0}^{z=4-2x-y} xz \, dz \, dy \, dx}$$



$2x + y + z = 4$

$@z=0 \rightarrow 2x + y = 4$

$y = 4 - 2x$

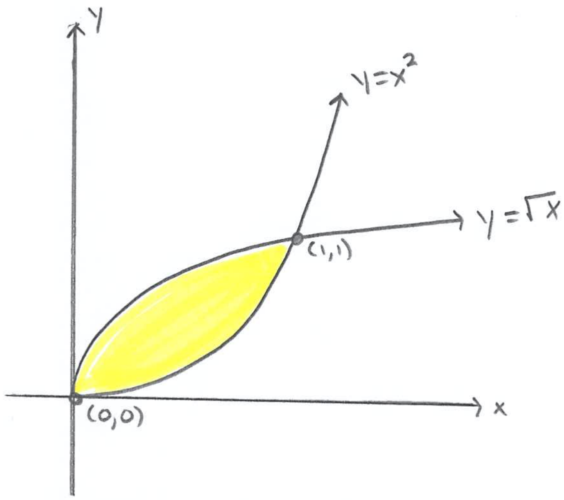


Problem 15

First notice that we want the region between $z = 3 - x^2 - y^2$ and above the region in the xy-plane ($z = 0$). This means we immediately know that:

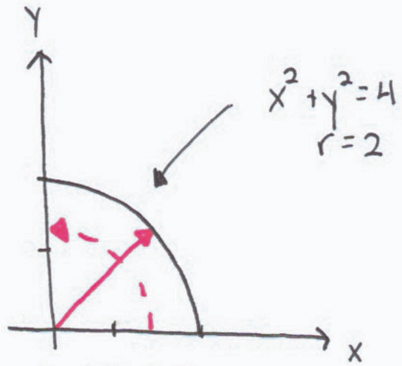
$$0 \leq z \leq 3 - x^2 - y^2$$

This gives us the bounds for our outermost integral, so we can now just look at the xy plane to set up the rest of the integral:



$$x^2 \leq y \leq \sqrt{x}$$
$$0 \leq x \leq 1$$

$$\boxed{\int_{x=0}^{x=1} \int_{y=x^2}^{y=\sqrt{x}} \int_{z=0}^{z=3-x^2-y^2} e^{xy} \, dz \, dy \, dx}$$



$0 \le r \le 2$

$0 \le \theta \le \pi/2$

$$\iint_D e^{\overbrace{-x^2-y^2}^{\to -r^2}}\, dA$$

$$= \iint_D e^{-r^2} \cdot r\, dr\, d\theta$$

$$= \left(\int_0^2 r e^{-r^2}\, dr \right)\left(\int_0^{\pi/2} 1 \; d\theta \right)$$

$u = -r^2$

$du = -2r\,dr$

$\int -\frac{1}{2} e^u\, du$

$-\frac{1}{2} e^{-r^2}$

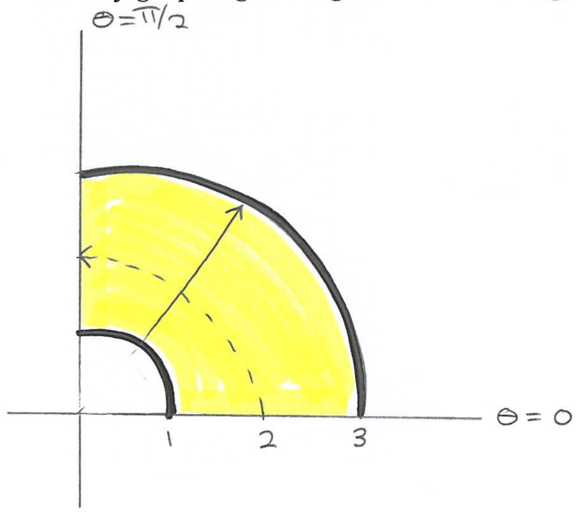
$$= \left(-\frac{1}{2} e^{-r^2} \Big]_0^2 \right)\left(\theta \Big]_0^{\pi/2} \right)$$

$$= \left(-\frac{1}{2} e^{-4} + \frac{1}{2} e^0 \right)\left(\frac{\pi}{2} - 0 \right)$$

$$= \left(\frac{1 - e^{-4}}{2} \right) \frac{\pi}{2} = \boxed{\frac{\pi(1-e^{-4})}{4}}$$

Problem 14

Start by graphing the region. Since the region is circular, use polar coordinates.



$$x^2 + y^2 = 1 \longrightarrow r = 1$$

$$x^2 + y^2 = 3 \longrightarrow r = 3$$

$$\frac{y^2}{x^2+y^2} = \frac{r^2\sin^2\theta}{r^2} = \sin^2\theta$$

$$\int_{\theta=0}^{\theta=\pi/2} \int_{r=1}^{r=3} \sin^2\theta \; r \, dr \, d\theta$$

$$\int_{\theta=0}^{\theta=\pi/2} \left. \frac{1}{2} r^2 \sin^2\theta \right]_{r=1}^{r=3} d\theta \longrightarrow \frac{1}{2} \cdot 9 \sin^2\theta - \frac{1}{2}\sin^2\theta = 4\sin^2\theta$$

$$\int_{\theta=0}^{\theta=\pi/2} 4\sin^2\theta \, d\theta = 4\int_{\theta=0}^{\theta=\pi/2} \frac{1}{2}(1-\cos(2\theta))\, d\theta$$

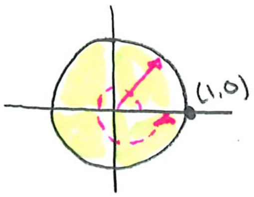
$$= 2\left(\theta - \frac{1}{2}\sin(2\theta)\right) = \left. 2\theta - \sin(2\theta) \right]_{\theta=0}^{\theta=\pi/2}$$

$$= \left[2\left(\frac{\pi}{2}\right) - \sin\left(\frac{2\pi}{2}\right)\right] - [0 - 0]$$

$$= \pi - 0$$

$$= \boxed{\pi}$$

Problem 15
Use the formulas for mass and center of mass:



$0 \le r \le 1$

$0 \le \theta \le 2\pi$

x-sym, y-sym.

$$m = \iint_D (4 + \cancel{2x}^{0} + \cancel{y}^{0})\, dA$$

$$= 4(\text{area of circle})$$

$$= 4\pi(1)^2$$

$$= \boxed{4\pi}$$

$$\bar{x} = \frac{1}{m}\iint_D x(4 + 2x + y)\, dA = \frac{1}{4\pi}\iint_D (\cancel{4x}^{0} + 2x^2 + \cancel{xy}^{0})\, dA = \frac{1}{4\pi}\iint_D 2x^2\, dA$$

$$= \frac{1}{4\pi}\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} 2r^2\cos^2\theta \, r\, dr\, d\theta = \frac{1}{4\pi}\left(\int_{r=0}^{r=1} 2r^3\, dr\right)\left(\int_{\theta=0}^{\theta=2\pi} \underbrace{\cos^2\theta}_{\frac{1}{2}(1+\cos 2\theta)}\, d\theta\right)$$

$$= \frac{1}{4\pi}\left(\frac{1}{2}r^4\Big]_0^1\right)\left(\frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi}\right)$$

$$= \frac{1}{4\pi}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left((2\pi + 0) - (0 + 0)\right) = \frac{1}{4\pi}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(2\pi) = \frac{1}{8}$$

$$\bar{y} = \frac{1}{m}\iint_D y(4 + 2x + y)\, dA = \frac{1}{4\pi}\iint_D (\cancel{4y}^{0} + \cancel{2xy}^{0} + y^2)\, dA$$

$$= \frac{1}{4\pi}\int_{\theta=0}^{\theta=2\pi} r^2\sin^2\theta\, r\, dr\, d\theta$$

$$= \frac{1}{4\pi}\left(\int_0^1 r^3\, dr\right)\left(\int_0^{2\pi} \underbrace{\sin^2\theta}_{\frac{1}{2}(1-\cos(2\theta))}\, d\theta\right)$$

$$= \frac{1}{4\pi}\left(\frac{1}{4}r^4\Big]_0^1\right)\left(\frac{1}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi}\right) = \frac{1}{4\pi}\left(\frac{1}{4}\right)\left(\frac{1}{2}(2\pi)\right) = \frac{1}{16}$$

$$\boxed{\text{C.O.M. } \left(\frac{1}{8}, \frac{1}{16}\right)}$$