MATH 220 Final Exam – Sample Test – Detailed Solutions

Problem 1

When $\lambda = 3$:

$$\begin{bmatrix} -2 & 2 & 2 \\ 3 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

This row reduces to:

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = x_2 + x_3$$

$$x_2 = 2x_3$$

$$x_3 = free$$

$$x_1 = 2x_3 + x_3 = 3x_3$$

$$x_2 = 2x_3$$

$$x_3 = free$$

So $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ (or any multiple of it) would be an appropriate eigenvector.

Problem 2
When $\lambda = 3$:

$$\begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$$

This row reduces to:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = -2x_2 - 3x_3$$

$$x_2 = free$$

$$x_3 = free$$

$$x_1 = -2x_2 - 3x_3$$

$$x_2 = 1x_2 + 0x_3$$

$$x_3 = 0x_2 + 1x_3$$

This gives a basis of $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Problem 3
When $\lambda = 1$:

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has two free variables, so there will be 2 linearly independent eigenvector, and the therefore the eigenspace will have a dimension of 2.

Problem 4

$$\begin{bmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 4 & 0 & -\lambda \end{bmatrix}$$

Doing a cofactor expansion down the second column gives us:

$$(2-\lambda)\begin{vmatrix} -\lambda & 1 \\ 4 & -\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(\lambda^2 - 4) = 0$$

$$(2-\lambda)(\lambda-2)(\lambda+2) = 0$$

$$\lambda = 2, \lambda = 2, \lambda = -2$$

Problem 5

First solve for the eigenvalues to get matrix D:

$$\begin{bmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{bmatrix}$$

$$(2-\lambda)(2-\lambda) - 49 = 0$$

$$4 - 4\lambda + \lambda^2 - 49 = 0$$

$$\lambda^2 - 4\lambda - 45 = 0$$

$$(\lambda - 9)(\lambda + 5) = 0$$

$$\lambda = 9, \lambda = -5$$

$$D = \begin{bmatrix} 9 & 0 \\ 0 & -5 \end{bmatrix}$$

Now find an eigenvector for each eigenvalue:

For $\lambda = 9$

$$\left[\begin{array}{cc|c} -7 & 7 & 0 \\ 7 & -7 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 = x_2$$

So we can choose $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as the eigenvector.

For $\lambda = -5$

$$\left[\begin{array}{cc|c} 7 & 7 & 0 \\ 7 & 7 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 = -x_2$$

So we can choose $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as the eigenvector giving us

$$D = \begin{bmatrix} 9 & 0 \\ 0 & -5 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Problem 6
Start by finding the eigenvalues:

Since this is a triangular matrix, we know the eigenvalues are the entries along the main diagonal. So the eigenvalue is $\lambda = 1$ with a multiplicity of 3.

Since we do not have n distinct eigenvalues, we need to find the eigenvector(s) corresponding to $\lambda = 1$. If there are 3 linearly independent eigenvectors, then it will be diagonalizable. If there are less than 3, it will not be diagonalizable.

For $\lambda = 1$:

$$\left[\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here there is only one free variable, x_1, so there will only be one linearly independent eigenvector, and therefore the matrix is not diagonalizable.

Problem 7
Here we are given the eigenvalues. Since we have 2 eigenvalues, and $n = 3$, we need to find the eigenvectors of each eigenvalue to see if the matrix is diagonalizable.

For $\lambda = 5$:

$$\left[\begin{array}{ccc|c} -3 & 2 & -1 & 0 \\ 1 & -2 & -1 & 0 \\ -1 & -2 & -3 & 0 \end{array}\right]$$

This row reduces to:

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = 2x_2 + x_3$$
$$x_2 = -x_3$$
$$x_3 = free$$

$$x_1 = 2(-x_3) + x_3 = -x_3$$
$$x_2 = -x_3$$
$$x_3 = free$$

Giving us an eigenvector of $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

For $\lambda = 1$:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 1 & 2 & -1 & 0 \\ -1 & -2 & 1 & 0 \end{array}\right]$$

This row reduces to:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = -2x_2 + x_3$$
$$x_2 = free$$
$$x_3 = free$$

$$x_1 = -2x_2 + 1x_3$$
$$x_2 = 1x_2 + 0x_3$$
$$x_3 = 0x_2 + 1x_3$$

Giving us an eigenvectors of $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Since the number of linearly independent eigenvectors matches the multiplicity of each corresponding eigenvalue this matrix is diagonalizable with

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P = \begin{bmatrix} -1 & -2 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Problem 8

Use the property that $A^3 = PD^3P^{-1}$.

First find P^{-1} using the formula

$$P^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$P^{-1} = \frac{1}{3-2}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$A^3 = PD^3P^{-1}$$

$$A^3 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

Multiply the first two matrices gives us:

$$A^3 = \begin{bmatrix} 24 & -2 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

Continuing to multiply the resulting matrix gives us:

$$A^3 = \begin{bmatrix} 26 & -54 \\ 9 & -19 \end{bmatrix}$$

Problem 9

a) Find the closest point is the same thing as finding $\hat{\mathbf{y}}$:

$$\hat{\mathbf{y}} = \left(\frac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 + \left(\frac{\mathbf{y}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\right)\mathbf{v}_2$$

$$\hat{\mathbf{y}} = \left(\frac{\frac{8}{3}+\frac{8}{3}+\frac{2}{3}}{\frac{4}{9}+\frac{1}{9}+\frac{4}{9}}\right)\mathbf{v}_1 + \left(\frac{\frac{-8}{3}+\frac{16}{3}+\frac{1}{3}}{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}}\right)\mathbf{v}_2$$

$$\hat{\mathbf{y}} = \left(\frac{\frac{18}{3}}{\frac{9}{9}}\right)\mathbf{v}_1 + \left(\frac{\frac{9}{3}}{\frac{9}{9}}\right)\mathbf{v}_2$$

$$\hat{\mathbf{y}} = (6)\mathbf{v}_1 + (3)\mathbf{v}_2$$

$$\hat{\mathbf{y}} = (6)\begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix} + (3)\begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix}$$

$$\hat{\mathbf{y}} = \begin{bmatrix}4\\2\\4\end{bmatrix} + \begin{bmatrix}-2\\2\\1\end{bmatrix} = \begin{bmatrix}2\\4\\5\end{bmatrix}$$

b) We already found $\hat{\mathbf{y}}$. To find the distance compute: $||\mathbf{y}-\hat{\mathbf{y}}||$

$$\mathbf{y}-\hat{\mathbf{y}} = \begin{bmatrix}4\\8\\1\end{bmatrix} - \begin{bmatrix}2\\4\\5\end{bmatrix} = \begin{bmatrix}2\\4\\-4\end{bmatrix}$$

$$||\mathbf{y}-\hat{\mathbf{y}}|| = \sqrt{4+16+16} = 6$$

Problem 10

To find the unit vector use the formula:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

$$\|\mathbf{v}\| = \sqrt{\left(\frac{2}{3}\right)^2 + (1)^2 + (0)^2 + (-2)^2}$$

$$\|\mathbf{v}\| = \sqrt{\frac{4}{9} + 1 + 4} = \sqrt{\frac{4}{9} + \frac{9}{9} + \frac{36}{9}} = \sqrt{\frac{49}{9}} = \frac{7}{3}$$

This give us:

$$\mathbf{u} = \frac{\mathbf{v}}{7/3} = \frac{3}{7}\mathbf{v} = \frac{3}{7}\begin{bmatrix} 2/3 \\ 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2/7 \\ 3/7 \\ 0 \\ -6/7 \end{bmatrix}$$

Problem 11

a) Set T is orthogonal because each vector dotted with any other vector equals 0.
b) Set T is not orthonormal because the length of each vector does not equal 1.
c) Set T is not a basis because it contains the 0 vector

Problem 12

$$c_1 = \frac{\mathbf{x} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} = \frac{8 + 0 - 3}{1 + 0 + 1} = \frac{5}{2}$$

$$c_2 = \frac{\mathbf{x} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}} = \frac{-8 - 16 - 3}{1 + 16 + 1} = -\frac{27}{18} = -\frac{3}{2}$$

$$c_3 = \frac{\mathbf{x} \cdot \mathbf{v_3}}{\mathbf{v_3} \cdot \mathbf{v_3}} = \frac{16 - 4 + 6}{4 + 1 + 4} = \frac{18}{9} = 2$$

Problem 13

$$\hat{\mathbf{y}} = \left(\frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u}$$

$$\hat{\mathbf{y}} = \left(\frac{5 - 1 + 8}{1 + 1 + 4}\right) \mathbf{u} = 2\mathbf{u} = 2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

Problem 14

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

$$\hat{\mathbf{y}} = \left(\frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u} = \left(\frac{28 + 12}{16 + 4}\right) \mathbf{u} = 2\mathbf{u} = 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Problem 15

Distance: $\|\mathbf{y} - \hat{\mathbf{y}}\|$

$$\hat{\mathbf{y}} = \left(\frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u} = \left(\frac{5 + 12 - 3 + 16}{1 + 9 + 1 + 4}\right)\mathbf{u} = 2\mathbf{u} = 2\begin{bmatrix} 1 \\ 3 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -2 \\ -4 \end{bmatrix}$$

$$\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ -8 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 5 \\ -4 \end{bmatrix}$$

$$\|\mathbf{y} - \hat{\mathbf{y}}\| = \sqrt{9 + 4 + 25 + 16} = \sqrt{54} = 3\sqrt{6}$$

Problem 16

$$\hat{\mathbf{y}} = \left(\frac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 + \left(\frac{\mathbf{y}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\right)\mathbf{v}_2$$

$$\hat{\mathbf{y}} = \left(\frac{-4+9+3+2}{1+9+1+4}\right)\mathbf{v}_1 + \left(\frac{-4+0+3-1}{1+1+1}\right)\mathbf{v}_2$$

$$\hat{\mathbf{y}} = \left(\frac{2}{3}\right)\mathbf{v}_1 + \left(\frac{-2}{3}\right)\mathbf{v}_2$$

$$\hat{\mathbf{y}} = \left(\frac{2}{3}\right)\begin{bmatrix}-1\\3\\1\\-2\end{bmatrix} + \left(\frac{-2}{3}\right)\begin{bmatrix}-1\\0\\1\\1\end{bmatrix}$$

$$\hat{\mathbf{y}} = \begin{bmatrix}-2/3\\2\\2/3\\-4/3\end{bmatrix} + \begin{bmatrix}2/3\\0\\-2/3\\-2/3\end{bmatrix} = \begin{bmatrix}0\\2\\0\\-2\end{bmatrix}$$

Problem 17

Use the Gram-Schmidt formulas where each column of the matrix represents $\mathbf{x_1}, \mathbf{x_2}$, and $\mathbf{x_3}$, respectively. The formulas are:

$$\mathbf{v_1} = \mathbf{x_1}$$

$$\mathbf{v_2} = \mathbf{x_2} - \left(\frac{\mathbf{x_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right)\mathbf{v_1}$$

$$\mathbf{v_3} = \mathbf{x_3} - \left(\frac{\mathbf{x_3} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right)\mathbf{v_1} - \left(\frac{\mathbf{x_3} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}}\right)\mathbf{v_2}$$

This gives us:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix} - \left(\frac{1 - 1 + 3 + 1}{1 + 1 + 1 + 1}\right)\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix} - (1)\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\mathbf{v_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{0 + 0 + 0 - 1}{1 + 1 + 1 + 1}\right)\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} - (0)\begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \left(\frac{1}{4}\right)\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 \\ -1/4 \\ 1/4 \\ 3/4 \end{bmatrix}$$

Note that in the answers, $\mathbf{v_3}$ is multiplied by 4 to get $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}$. This is okay to do since scaling a vector by a constant, will not change whether or not it is orthogonal to the other vectors. So this still forms an orthogonal basis.

Problem 18

Since the vectors are not orthogonal, use the Gram-Schmidt process to find an orthogonal basis, then divide each vector by its length to make them unit vectors.

The formulas are:

$$\mathbf{v_1} = \mathbf{x_1}$$

$$\mathbf{v_2} = \mathbf{x_2} - \left(\frac{\mathbf{x_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right)\mathbf{v_1}$$

This gives us:

$$\mathbf{v_1} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$\mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \left(\frac{3 + 12 + 0}{9 + 36 + 0}\right)\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

So the orthogonal basis is: $\left\{\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right\}$

Now divide each vector by its length to get an orthonormal basis:

$$\|\mathbf{v_1}\| = \sqrt{9 + 36 + 0} = 3\sqrt{5}$$

$$\|\mathbf{v_2}\| = \sqrt{0 + 0 + 4} = 2$$

Orthonormal basis:

$$\left\{\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

<u>Problem 19</u> PLEASE NOTE THERE IS AN ERROR ON THIS PROBLEM! IT SHOULD READ

$$\text{Find a least squares solution of } A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}$$

$$A^T \mathbf{b} = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & -11 & -4 \\ -11 & 22 & 11 \end{array}\right] \sim \left[\begin{array}{cc|c} 6 & -11 & -4 \\ 1 & -2 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 6 & -11 & -4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

$$\hat{\mathbf{x}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Problem 20

$$A^T A = \begin{bmatrix} 0 & 0 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} 0 & -5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 25 \end{bmatrix}$$

$$\lambda_1 = 25, \lambda_2 = 0$$

$$\sigma_1 = 5, \sigma_2 = 0$$

Matrix D: [5]

$$\text{Matrix } \Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$$

For $\lambda_1 = 25$: $\left[\begin{array}{cc|c} -25 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$. So $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

For $\lambda_2 = 0$: $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 25 & 0 \end{array}\right]$. So $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ (Remember that matrix V must be orthogonal so choose your free variables appropriately)

So Matrix V = $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $V^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\mathbf{u_1} = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{5}\begin{bmatrix} 0 & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -5 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Because $\sigma_2 = 0$ cannot find $\mathbf{u_2}$ algebraically like we did for $\mathbf{u_1}$. However, remember that matrix U must be orthonormal so choose any vector for $\mathbf{u_2}$ with a length of 1 that is orthogonal to $\mathbf{u_1}$.

$$\mathbf{u_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So Matrix U = $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

SVD: $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$