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MATH220 Exam 1 – Sample Test 1 – Detailed Solutions

Problem 1 (B)

$$A = \begin{bmatrix} \boxed{2} & 0 & 5 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & \boxed{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \boxed{1} & -2 & 0 & -1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}, \quad C = \begin{bmatrix} \boxed{1} & 5 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

EF **RREF** **NEITHER**

Problem 2 (A)

Get the matrix into reduced row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -5 \\ 2 & 3 & -1 & -2 \\ 2 & -1 & 0 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -5 \\ 0 & 1 & 3 & 8 \\ 0 & -3 & 4 & 15 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -5 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 13 & 39 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -5 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -5 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & -13 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

So $x_1 = 2, x_2 = -1, x_3 = 3$

Problem 3 (A)

Put the matrix in echelon form. In order to have infinity many solutions we want a row of 0's on bottom:

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & h & k \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & h+6 & k-2 \end{bmatrix}$$

So $h + 6 = 0$ and $k - 2 = 0$, which gives us $h = -6, k = 2$.

Problem 4 (D)

For b to be in the span of $\mathbf{a_1}$ and $\mathbf{a_2}$ the following matrix must be consistent:

$$\left[\begin{array}{cc|c} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{array}\right]$$

Put it in echelon form:

$$\left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & h+8 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 3 & h+8 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & h+17 \end{array}\right]$$

Since the coefficient matrix has a row of 0's on bottom, then to be consistent $h + 17$ must equal 0.

$$h = -17$$

Problem 5 (C)
It's helpful to draw out 3 possibilities and see what works:

$$n > m \qquad \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}$$

$$n = m \qquad \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}$$

$$n < m \qquad \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}$$

I. True. It is not possible to have a pivot in every column if $n < m$, so $n \geq m$

II. True. Since there is a pivot in every column, there are no free variables, which means $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

III. False. Since there is a pivot in every column, the vectors are linearly independent, which means they span $\mathbb{R}^m$

Problem 6 (C)
Solve the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & 1 & -4 & 0 \\ -1 & 0 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = 2x_3 - x_2$$
$$x_2 = 0$$
$$x_3 = \text{free}$$

Choose anything you want for x_3, and find x_1 based on that. For example if $x_3 = 1$, then

$$x_1 = 2(1)$$
$$x_2 = 0$$
$$x_3 = 1$$

So $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ or any multiple of $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ would be a solution.

Problem 7 (A)

Since we only have one equation, and 3 variables, then one variable must depend on the other two…

Let x_2 and x_3 be free variables…

$$\begin{aligned} x_1 &= 2 - 3x_2 + 1x_3 \\ x_2 &= 0 + 1x_2 + 0x_3 \\ x_3 &= 0 + 0x_2 + 1x_3 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3$$

Problem 8 (B)

a) Not linearly independent because it contains the 0 vector
b) Linearly Independent (You could double check this by finding the echelon form of the matrix and showing that there are no free variables)
c) Not linearly independent because there are more columns than rows
d) Not linearly independent because $\mathbf{v_2} - \mathbf{v_1} = \mathbf{v_3}$

Problem 9 (D)

$$\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}$$

Notice that it is impossible to have a pivot in every row. So according to our theorem since one of these conditions is false, **they must all be false**:

1. $A\mathbf{x}=\mathbf{b}$ has a solution for all **b** in $\mathbb{R}^m$
2. Every **b** in $\mathbb{R}^m$ is a linear combination of the columns in A
3. The columns of A span $\mathbb{R}^m$
4. A has a pivot in every row

Looking at condition 3, tells us that answer choice D is therefore true.

Problem 10 (D)

For linear dependence, we want the matrix associated with vectors to have free variables. Put it in echelon form:

$$\begin{bmatrix} 1 & -2 & 3 \\ 5 & -9 & h \\ -3 & 6 & -9 \end{bmatrix}$$

$$\frac{1}{3}R_3$$

$$\begin{bmatrix} 1 & -2 & 3 \\ 5 & -9 & h \\ -1 & 2 & -3 \end{bmatrix}$$

$$-5R_1 + R_2, R_1 + R_3$$

$$\begin{bmatrix} \boxed{1} & -2 & 3 \\ 0 & \boxed{1} & h - 15 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there are only 2 pivot columns, there will be a free variable no matter what h is. Therefore these vectors are linearly dependent for all values of h.

Problem 11 (C)

$$A = \begin{bmatrix} \boxed{1} & 0 & 3 & 4 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \boxed{1} & 0 & 4 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}, \quad C = \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{-1} \end{bmatrix}$$

EF   RREF   EF

Problem 12 (B)

First consider the bottom row: Since we have a row of 0's on the bottom, then for the matrix to be consistent $a - 1$ must equal 0. So $a = 1$.

Next consider the middle row: Since have a 0 in the right column, there is no restriction on b.

Problem 13 (B)

a) Is in the span because it is one of the vectors itself.
b) You can choose this answer by process of elimination, or you can set up the matrix below and show that it is inconsistent when reduced to echelon form.

$$\begin{bmatrix} 1 & 4 & 1 \\ 2 & 5 & 6 \\ 3 & 6 & 7 \end{bmatrix}$$

c) Is in the span because it is $\mathbf{v_1} + \mathbf{v_2}$
d) Is in the span because it is 0 times either vector.

Problem 14 (D)
This matrix already in echelon form:

$$\begin{bmatrix} \boxed{2} & 1 & 5 & 0 & 5 \\ 0 & \boxed{3} & 7 & 0 & 1 \\ 0 & 0 & 0 & \boxed{3} & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There is a column without a pivot so there is a free variable. Therefore there are infinitely many solutions.

Problem 15 (D)
For b to be in the span of $\mathbf{a_1}$ and $\mathbf{a_2}$ the following matrix must be consistent:

$$\left[\begin{array}{cc|c} 1 & -3 & -3 \\ 4 & -7 & 3 \\ -2 & 5 & h \end{array}\right]$$

Put it in echelon form: $-4R_1 + R_2, 2R_1 + R_3$

$$\left[\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & 5 & 15 \\ 0 & -1 & h-6 \end{array}\right]$$

$\frac{1}{5}R_2$

$$\left[\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & 1 & 3 \\ 0 & -1 & h-6 \end{array}\right]$$

$R_2 + R_3$

$$\left[\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & h-3 \end{array}\right]$$

Since the coefficient matrix has a row of 0's on bottom, to be consistent $h - 3$ must equal 0.

$$h = 3$$

Problem 16 (A)

Since we only have one equation, and 3 variables, then one variable must depend on the other two…

Let x_2 and x_3 be free variables…

$$\begin{aligned} x_1 &= 3 + 7x_2 - 1x_3 \\ x_2 &= 0 + 1x_2 + 0x_3 \\ x_3 &= 0 + 0x_2 + 1x_3 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} x_3$$

Problem 17 (D)

a) Not LI because it contains the 0 vector.
b) Not LI because $\mathbf{v_1} + \mathbf{v_2} = \mathbf{v_3}$
c) Not LI because $(-1)\mathbf{v_1} = \mathbf{v_3}$
d) **Linearly Independent.** You can choose this answer by process of elimination or by setting up the matrix and showing that when reduced to echelon form there are no free variables.

Problem 18 (B)

Linearly independent means there is a pivot in every column. Since a 5×3 matrix has 3 columns, there will be 3 pivot columns.

Problem 19 (D)

If $A \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, that means that $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution, which by definition means the columns of A are linearly dependent.

This also means there is NOT a pivot in every column, which makes answer choice C false.

Also note that since A is a 4×3 matrix, there must be at least one row without a pivot. Because of this fact we can use the Theorem that says that answer choices A and B will also be false.

Problem 20 (D)

Set up a matrix and see how many linearly independent vectors are in the span (how many pivot columns):

$$\begin{bmatrix} 1 & 7 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

Notice this matrix is already in echelon form, and we can see there are 3 pivots, so there are linearly independent vectors. Therefore the span will have a dimension of 3. In other words, it will take up the whole space $\mathbb{R}^3$.