MATH141 Exam 1 – Sample Test – Detailed Solutions

Problem 1: C
Use u-substitution:

$u = 1 - e^x$

$du = -e^x dx$

$dx = \frac{du}{-e^x}$

$$\int \frac{e^x}{(1-e^x)^2} dx$$

$$\int \frac{e^x}{u^2} \frac{du}{-e^{-x}}$$

$$-\int \frac{1}{u^2} du$$

$$-\left(\frac{u^{-1}}{-1}\right) + C$$

$$\frac{1}{u} + C$$

$$\frac{1}{1-e^x} + C$$

Problem 2: B
Use u-substitution and recognize this as a $\tan^{-1}$ integral:

$u = x^2$

$du = 2xdx$

$dx = \frac{du}{2x}$

$$\int \frac{x}{1+x^4} dx$$

$$\int \frac{x}{1+(x^2)^2} dx$$

$$\int \frac{x}{1+u^2} \frac{du}{2x}$$

$$\frac{1}{2} \int \frac{1}{1+u^2} du$$

$$\frac{1}{2} \tan^{-1} u + C$$

$$\frac{1}{2} \tan^{-1}(x^2) + C$$

Problem 3: E

Use u-substitution:

$$u = 1 + \ln x$$

$$du = \frac{1}{x} dx$$

$$dx = x du$$

$$\int_{-}^{-} \frac{1}{xu^2} x \, du$$

$$\int_{-}^{-} u^{-2} \, du$$

$$\frac{u^{-1}}{-1} = -\frac{1}{u} = -\frac{1}{1 + \ln x}\bigg]_{e}^{e^2}$$

$$\left[-\frac{1}{1 + \ln(e^2)}\right] - \left[-\frac{1}{1 + \ln(e)}\right]$$

$$\left[-\frac{1}{1 + 2}\right] - \left[-\frac{1}{1 + 1}\right]$$

$$-\frac{1}{3} + \frac{1}{2} = \frac{1}{6}$$

Problem 4: B

Use integration by parts

$$u = \tan^{-1} x \qquad v = x$$

$$du = \frac{1}{1+x^2}dx \qquad dv = dx$$

Applying the formula gives us:

$$uv - \int v\,du$$

$$x\tan^{-1} x - \int \frac{x}{1+x^2}dx$$

Use a u substitution to compute the remaining integral:

$$\int \frac{x}{1+x^2}dx \qquad u = 1 + x^2$$

$$= \int \frac{x}{u}\left(\frac{du}{2x}\right) \qquad du = 2xdx$$

$$dx = \frac{du}{2x}$$

$$= \frac{1}{2}\int \frac{1}{u}du$$

$$= \frac{1}{2}\ln u$$

$$= \frac{1}{2}\ln(1+x^2)$$

Now evaluate from 0 to 1:

$$\left[x\tan^{-1} x - \frac{1}{2}\ln(1+x^2)\right]_0^1$$

$$\left[\tan^{-1}(1) - \frac{1}{2}\ln(2)\right] - \left[0 - \frac{1}{2}\ln(0)\right]$$

$$\frac{\pi}{4} - \frac{1}{2}\ln 2$$

Problem 5: D
Since all powers or sine and cosine are even, use the half-angle identities:

$$\int_0^{\pi/8} \frac{1}{2}(1+\cos 2x)\,\frac{1}{2}(1-\cos 2x)\,dx$$

$$\frac{1}{4}\int_0^{\pi/8} 1-\cos^2(2x)\,dx$$

Use the half angle identity again:

$$\frac{1}{4}\int_0^{\pi/8} \left(1-\frac{1}{2}(1-\cos(4x)\right)dx$$

$$\frac{1}{4}\int_0^{\pi/8} \left(\frac{1}{2}-\frac{1}{2}\cos(4x)\right)dx$$

$$\frac{1}{4}\left[\frac{1}{2}x-\frac{1}{8}\sin(4x)\right]_0^{\pi/8}$$

$$\frac{1}{4}\left[\frac{\pi}{16}-\frac{1}{8}\sin\left(\frac{\pi}{2}\right)\right]-\frac{1}{4}\left[0-\frac{1}{8}\sin(0)\right]$$

$$\frac{1}{4}\left[\frac{\pi}{16}-\frac{1}{8}\right]=\frac{1}{4}\left[\frac{\pi-2}{16}\right]=\frac{\pi-2}{64}$$

Problem 6: A
Since $\sin x$ is odd, $u=\cos x$

$$\int \sin^3 x\, u^6 \frac{du}{-\sin x}$$

$$-\int \sin^2 x\, u^6\, du$$

$$-\int (1-u^2)u^6\, du$$

$$-\int (u^6-u^8)\, du$$

$$-\frac{1}{7}u^7+\frac{1}{9}u^9+C$$

$$\frac{1}{9}\sin^9 x-\frac{1}{7}\sin^7 x+C$$

$$u=\cos x$$

$$du=-\sin x\, dx$$

$$dx=\frac{du}{-\sin x}$$

$$\sin^2 x=1-\cos^2 x$$

$$\sin^2 x=1-u^2$$

Problem 7: E
Use integration by parts, twice.

Apply integration by parts with:

$$u = x^2 \qquad v = \frac{1}{2}\sin(2x)$$

$$du = 2xdx \qquad dv = \cos(2x)\,dx$$

$$uv - \int v\,du$$

$$\frac{1}{2}x^2\sin\,(2x) - \int \frac{1}{2}2x\sin(2x)\,dx$$

$$\frac{1}{2}x^2\sin\,(2x) - \int x\sin(2x)\,dx$$

Now apply integration by parts with:

$$u = x \qquad v = -\frac{1}{2}\cos(2x)$$

$$du = dx \qquad dv = \sin(2x)\,dx$$

$$\frac{1}{2}x^2\sin\,(2x) - \left[uv - \int v\,du\right]$$

$$\frac{1}{2}x^2\sin(2x) - \left[-\frac{1}{2}x\cos(2x) + \int \frac{1}{2}\cos(2x)\,dx\right]$$

$$\frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \int \frac{1}{2}\cos(2x)\,dx$$

$$\frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin\,(2x) + C$$

Problem 8: D

Use the substitution

$x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$

The bounds will change to…

$x = 0 \rightarrow\ 0 = 3\tan\theta$

$\tan\theta = 0$

$\theta = 0$

$x = 3 \rightarrow\ 3 = 3\tan\theta$

$\tan\theta = 1$

$\theta = \frac{\pi}{4}$

The integral becomes…

$$\int_0^{\pi/4} \frac{1}{9\tan^2\theta\, f\left(\sqrt{9\tan^2\theta + 9}\right)} 3\sec^2\theta\, d\theta$$

$$\int_0^{\pi/4} \frac{\sec^2\theta}{3\tan^2\theta\, f(3\sec\theta)} d\theta$$

$$\frac{\sec^2\theta}{\tan^2\theta} = \frac{\frac{1}{\cos^2\theta}}{\frac{\sin^2\theta}{\cos^2\theta}} = \csc^2\theta$$

$$\int_0^{\pi/4} \frac{\csc^2\theta}{3f(3\sec\theta)} d\theta$$

Problem 9: B

Use a u substitution:

$$\int f'(2x)e^u \frac{du}{2f'(2x)}$$

$$u = f(2x)$$

$$du = 2f'(2x)dx$$

$$dx = \frac{du}{2f'(2x)}$$

$$= \int \frac{1}{2} e^u \, du$$

$$= \frac{1}{2} e^u$$

$$= \frac{1}{2} e^{f(2x)} \Big]_0^2$$

$$= \frac{1}{2} \left[e^{f(4)} - e^{f(0)}\right]$$

$$= \frac{1}{2} [e^5 - e]$$

Problem 10: C

Use integration by parts:

$$u = 2x \qquad v = f'(x)$$

$$du = 2dx \qquad dv = f''(x)dx$$

Applying the formula gives us:

$$uv - \int v \, du$$

$$2xf'(x)|_0^4 - \int_0^4 2f'(x)dx$$

$$2xf'(x)|_0^4 - 2f(x)|_0^4$$

$$2[4f'(4) - 0f'(0)] - 2[f(4) - f(0)]$$

$$2[4(11) - 0] - 2[5 - 1]$$

$$88 - 8 = 80$$

Problem 11: E

$$\int_0^2 f'(4x)dx$$

$$= \frac{1}{4} f(4x)\Big|_0^2$$

$$= \frac{1}{4}[f(8) - f(0)]$$

$$= \frac{1}{4}[10 - 1] = \frac{9}{4}$$

Problem 12: B

If $y = Ae^{3x}$, then $y' = 3Ae^{3x}, y'' = 9Ae^{3x}$

Plug into $y'' + y' - 2y = 6e^{3x}$:

$$9Ae^{3x} + 3Ae^{3x} - 2Ae^{2x} = 6e^{3x}$$

$$10Ae^{3x} = 6e^{3x}$$

$$10A = 6$$

$$A = 3/5$$

Problem 13

In this problem you will first need to use a u-substitution, then integration by parts. To avoid using the variable u twice in one problem, I will use "t" instead of u in the first u-substiution.

$$\int e^t\, 2\sqrt{x}\, dt$$

$$t = \sqrt{x}$$

$$dt = \frac{1}{2\sqrt{x}}dx$$

$$dx = 2\sqrt{x}dt$$

Now to get everything in terms of t, we can sub in $\sqrt{x} = t$ to get:

$$\int 2te^t\, dt$$

Apply integration by parts:

$$u = 2t \qquad v = e^t$$

$$du = 2\, dt \qquad dv = e^t dt$$

$$\int 2te^t\, dt = 2te^t - \int 2e^t\, dt$$

$$= 2te^t - \int 2e^t\, dt$$

$$= 2te^t - 2e^t + C$$

Now plug back in $t = \sqrt{x}$ to get the final answer in terms of x:

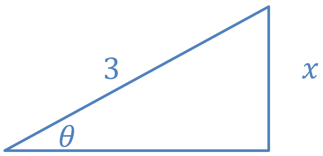
$$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

Problem 14

This is a trig sub integral.

$x = 3\sin\theta \rightarrow \sin\theta = \frac{x}{3} = \frac{opp}{hyp}$

$dx = 3\cos\theta\, d\theta$



$$\int_0^3 \frac{x^2}{\sqrt{9-x^2}}\,dx$$

$$\int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}3\cos\theta\,d\theta = \int \frac{9\sin^2\theta}{\sqrt{9(1-\sin^2\theta)}}3\cos\theta\,d\theta = \int \frac{9\sin^2\theta}{3\cos\theta}3\cos\theta\,d\theta$$

$$\int 9\sin^2\theta\,d\theta$$

$$9\int\left(\frac{1}{2}-\frac{1}{2}\cos(2\theta)\right)d\theta$$

$$9\left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]$$

$$9\left[\frac{1}{2}\theta - \frac{1}{4}(2\sin\theta\cos\theta)\right]$$

$$9\left[\frac{1}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right)\right]_0^3$$

$$9\left[\frac{1}{2}\sin^{-1}(1) - \frac{1}{2}(1)(0)\right] - 9\left[\frac{1}{2}\sin^{-1}(0) - \frac{1}{2}(0)(1)\right]$$

$$9\left[\frac{1}{2}\left(\frac{\pi}{2}\right) - 0\right] - 0$$

$$\frac{9\pi}{4}$$

Problem 15

This is a partial fraction integral.

$$\frac{2x-3}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

$$2x-3 = A(x^2+3) + (Bx+C)x$$

Let $x = 0$: $-3 = A(3) \rightarrow \ \boldsymbol{A = -1}$

$$2x-3 = -1(x^2+3) + (Bx+C)x$$

$$2x-3 = -x^2 - 3 + Bx^2 + Cx$$

$$2x-3 = (-1+B)x^2 + Cx - 3$$

This gives us:

$$\begin{cases} -1 + B = 0 \rightarrow \boldsymbol{B} = 1 \\ \boldsymbol{C = 2} \end{cases}$$

$$\int \frac{2x-3}{x^3+3x}\,dx = \int \left(-\frac{1}{x} + \frac{x+2}{x^2+3}\right) dx$$

$$\int \left(-\frac{1}{x} + \frac{x}{x^2+3} + \frac{2}{x^2+3}\right) dx$$

$$\int -\frac{1}{x}\,dx + \int \frac{x}{x^2+3}\,dx + \int \frac{2}{x^2+3}\,dx$$

$$-\ln x + \frac{1}{2}\ln(x^2+3) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

For $\int \frac{x}{x^2+3}\,dx$ use a $u - sub$, where $u = x^2+3$

For $\int \frac{2}{x^2+3}\,dx$ pull out the 2 and recognize this as an arctan integral

Problem 16

Integrate both sides….

$$\int (2y-6)dy = \int (2-e^x)dx$$

$$y^2 - 6y = 2x - e^x + c$$

Plug in $y(0) = 0$ to solve for C:

$$0 = -1 + C$$
$$C = 1$$

$$y^2 - 6y = 2x - e^x + 1$$

Complete the square to solve for y:

$$y^2 - 6y + \boxed{9} = 2x - e^x + 1 + \boxed{9}$$

$$(y-3)^2 = 2x - e^x + 10$$

$$y - 3 = \pm\sqrt{2x - e^x + 10}$$

$$y = 3 \pm \sqrt{2x - e^x + 10}$$

Since we have to meet the initial value of $y(0) = 0$, then $y = 3 - \sqrt{2x - e^x + 10}$ is the only solution.

Problem 17

$$\frac{dy}{dx} = x^2 + x^2y^2$$

$$dy = x^2(1+y^2)dx$$

Integrate both sides….

$$\int \frac{1}{1+y^2}dy = \int x^2 dx$$

$$\tan^{-1} y = \frac{1}{3}x^3 + C$$

Plug in $y(0) = 1$ to solve for C:

$$\tan^{-1}(1) = 0 + C$$
$$\frac{\pi}{4} = 0 + C$$
$$C = \frac{\pi}{4}$$

$$\tan^{-1} y = \frac{1}{3}x^3 + \frac{\pi}{4}$$

Solve for y:

$$y = \tan\left(\frac{1}{3}x^3 + \frac{\pi}{4}\right)$$

Problem 18

a) Yes, since there are only y's and no other variables.

b) Set y'=0

$$y^4 - 9y^2 = 0$$

$$y^2(y^2 - 9) = 0$$

$$y^2(y + 3)(y - 3) = 0$$

$$y = 0, y = -3, y = 3$$

c)

$y = 3$, unstable
$y = 0$, semistable
$y = -3$, stable

Problem 19

Use Newton's Law of Cooling

$$T_0 = 35, \qquad y(0) = 10, \qquad y(5) = 15$$

$y' = -k(y - T_0)$
$y' = -k(y - 35)$

$y = 35 + Ce^{-kt}$

Find C:
$10 = 35 + Ce^0$
$C = -25$

Find k:
$y = 35 - 25e^{-kt}$
$15 = 35 - 25e^{-5k}$
$-20 = -25e^{-5k}$
$4/5 = e^{-5k}$
$\ln\left(\frac{4}{5}\right) = -5k$
$k = \frac{\ln\left(\frac{4}{5}\right)}{-5}$

$y = 35 - 25e^{\frac{1}{5}\ln\left(\frac{4}{5}\right)t}$

Problem 20

a) $P' = 4 - 0.10P$

b) $0 = 4 - 0.10P$
$.10P = 4$
$P = \frac{4}{.10} = 40$

c) Solve part A as a separable differential equation:

$$\frac{dP}{dt} = 4 - 0.10P$$

$$\frac{1}{4 - 0.10P} dP = dt$$

$$\frac{\ln(4 - 0.10P)}{-0.10} = t + C$$

$$\ln(4 - 0.10P) = -0.10t + C$$

$$4 - 0.10P = Ce^{-0.10t}$$

$$0.10P = 4 - Ce^{-0.10t}$$

$$P = \frac{4 - Ce^{-0.10t}}{0.10}$$

$$P = 40 - Ce^{-0.10t}$$

Use $P(0) = 50$ to solve for C:

$50 = 40 - Ce^0$
$10 = -C$
$C = -10$

$$P = 40 + 10e^{-0.10t}$$

d) 40,000 fish